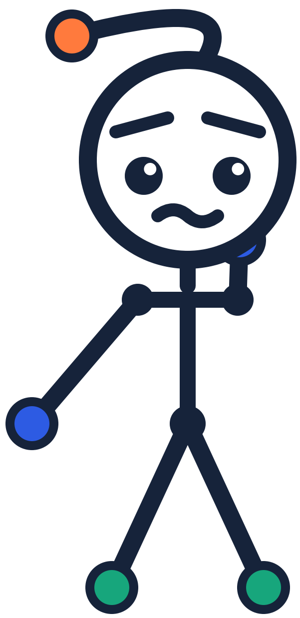
Young Innovators
FOR CHANGE
innovateyouth.org

A FULL-YEAR ENRICHMENT PROGRAM · GRADE 4

It Cannot Be Done

36 weekly assignments on proving things, including the things that *cannot happen*



This book belongs to

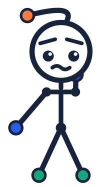
Teacher

School year

One assignment a week · two pages, front and back · about 30 to 40 minutes
Built to sit alongside the California Grade 4 math curriculum, not to replace it.

How This Book Works

Read this page with a grown-up before Week 1.



Hello again. I am Pip.

Grade 3 was about counting things properly. This year is about a harder move: showing that something **cannot happen at all**, without trying every case. Nine weeks from now you will settle a puzzle that beat a whole city for years, in three lines.

One assignment a week, two pages

Page 1 has a note about what you are doing in class this week, a two-minute **Number Warm-Up**, the **Words to keep**, and the main problem.

Page 2 has more to try, then **Add to your Proof File**, a **Talk About It** question, a hint, and a ★ **Challenge Zone**.

The Proof File

One folder, built all year, one page per proof. Each page has a **claim** and a **reason**, never just an answer. By Week 36 it is about thirty-three pages and it is the part you keep. Get the folder in Week 1.

Three rules for this book

1. **I tried and could not** is not a proof. It is a report on your afternoon.
2. To knock a claim down you need **one** example. To hold it up you need a reason that covers every case.
3. When a reason works, stop looking for more examples. That is what a reason is for.

For the grown-up helping

The question worth repeating all year is **“how do you know that covers every case?”** Children arrive in Grade 4 believing that enough examples settle a question. Most of this book is the slow correction of that belief, and it takes the whole year, so do not rush a page where the answer is already obvious to them. The answer is never the point here. The reason is.

The Year at a Glance

Weeks 1–18. The last column is what the class is usually doing that week.

WK	ASSIGNMENT	STRAND	IN CLASS THIS WEEK
TRIMESTER 1 · WEEKS 1–12			
1	**The Thing That Never Changes**	Logic & Proof	Number and shape patterns (4.OA.5)
2	**Odd, Even and Impossible**	Number Structure	Number patterns; place value (4.OA.5, 4.NBT.4)
3	**Factor Rectangles**	Number Structure	Factors, multiples, prime and composite (4.OA.4)
4	**Primes and Composites**	Number Structure	Prime and composite numbers (4.OA.4)
5	**Building From Primes**	Number Structure	Factors; multi-digit multiplication (4.OA.4, 4.NBT.5)
6	**Multiples That Meet**	Number Structure	Factors and multiples (4.OA.4)
7	**The Biggest Shared Factor**	Number Structure	Factors and multiples (4.OA.4)
8	**Coloring Proves It**	Logic & Proof	Multi-step problems and reasoning (4.OA.3)
9	**The Domino Problem**	Logic & Proof	Multi-step problems and reasoning (4.OA.3)
10	**Degree Tells You**	Dots & Lines	Multi-step problems and reasoning (4.OA.3)
11	**Trace It or Prove You Cannot**	Dots & Lines	Multi-step problems and reasoning (4.OA.3)
12	**Impossible Fair**	Logic & Proof	Multi-step problems and reasoning (4.OA.3)
TRIMESTER 2 · WEEKS 13–24			
13	**Case Files**	Logic & Proof	Review & reasoning
14	**Divisible by 2, 5 and 10**	Number Structure	Place value to one million (4.NBT.1, 4.NBT.2)
15	**Divisible by 3 and 9**	Number Structure	Place value and multi-digit arithmetic (4.NBT.1, 4.NBT.4)
16	**Casting Out Nines**	Number Structure	Multi-digit multiplication (4.NBT.5)
17	**Wrapping Around, Again**	Number Structure	Multi-digit arithmetic and patterns (4.NBT.4, 4.OA.5)
18	**The Day of the Week**	Number Structure	Multi-step problems; measurement (4.OA.3, 4.MD.2)

The Year at a Glance

Weeks 19–36. The last column is what the class is usually doing that week.

WK	ASSIGNMENT	STRAND	IN CLASS THIS WEEK
19	**Lines of Symmetry**	Shape & Space	Lines of symmetry (4.G.3)
20	**Turning a Shape**	Shape & Space	Lines of symmetry; angle measure (4.G.3, 4.MD.5)
21	**The Eight Ways to Move a Square**	Shape & Space	Angle measure and reasoning (4.MD.5, 4.OA.3)
22	**Angles That Must Add Up**	Shape & Space	Angle measure and adding angles (4.MD.5, 4.MD.7)
23	**Tiling the Floor**	Shape & Space	Angle measure (4.MD.5, 4.MD.7)
24	**Impossible Tilings**	Shape & Space	Multi-step problems and reasoning (4.OA.3)
TRIMESTER 3 · WEEKS 25–36			
25	**Puzzle Fair**	Logic & Proof	Review & reasoning
26	**Choosing a Team**	Ways of Counting	Multi-step problems (4.OA.3)
27	**The Triangle Grows**	Ways of Counting	Number and shape patterns (4.OA.5)
28	**The Triangle and the Teams**	Ways of Counting	Number and shape patterns (4.OA.5)
29	**Trees**	Dots & Lines	Multi-step problems and reasoning (4.OA.3)
30	**The Cheapest Network**	Dots & Lines	Multi-step problems (4.OA.3)
31	**Two Ways to Multiply**	Steps & Rules	Multiplying multi-digit numbers (4.NBT.5)
32	**The Euclid Machine**	Steps & Rules	Multi-step problems; factors (4.OA.3, 4.OA.4)
33	**The Same Point on the Line**	Fractions	Equivalent fractions (4.NF.1, 4.NF.2)
34	**Parts of a Hundred**	Fractions	Decimal notation for fractions (4.NF.5, 4.NF.6)
35	**The Weighing Puzzles**	Logic & Proof	Multi-step problems and reasoning (4.OA.3)
36	**The Proof File, Finished**	Logic & Proof	Review of the year

WEEK 1 LOGIC & PROOF

The Thing That Never Changes

Find what a move cannot touch, and you can rule out the future.

IN CLASS THIS WEEK
Number and shape patterns (4.OA.5)

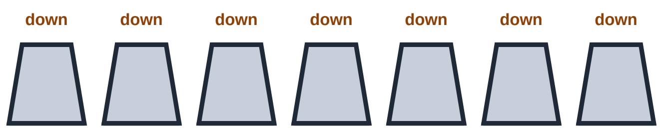
NUMBER WARM-UP · 2 MINUTES
Is 27 odd or even? Add 2. Still odd? Add 2 again.

WORDS TO KEEP

move one action the rules allow you to take

invariant something that stays the same no matter which moves you make, or how many

Seven cups, all upside down. The one rule: **turn over exactly two cups**, any two you like. Your goal is to get every cup the right way up.



the starting position

1 Watch one number.

a. How many cups are upside down at the start? ________

b. A move turns two cups. There are three kinds. Write what each one does to the count: does it go up by 2, down by 2, or stay the same?

two down cups turned up: ______________________

one down and one up: ______________________

two up cups turned down: ______________________

c. All three changes are ______________________ numbers, so the count can never flip between odd and even.

2 Now the goal.

a. To finish, how many cups must be upside down? ________ Is that odd or even? ________

b. You started odd, and every move keeps it odd. So can you ever finish? ________

c. How many moves did you have to try to be sure? ________

3 Six cups instead.

Same rule, but now **six** cups start upside down.

a. Is it possible now? ________

b. Show it. Write which two cups you turn on each move, and how many moves you need.

____

____

✎ ADD TO YOUR PROOF FILE

Start the Proof File. Page 1 is this claim, written out in three lines: what you cannot do, the number that never changes, and why that settles it.

💬 TALK ABOUT IT

You proved something about moves nobody has made yet. How is that different from trying for an hour and giving up?

The invariant here is not the number of upside-down cups. It is something about that number. Say exactly what.

STUCK? TRY THIS

Do not look for a clever sequence of moves. Look for a number that every single move leaves alone, and then check whether the goal has the right value of it.

★ CHALLENGE ZONE

Same seven cups, but now you must turn over exactly **three** at a time. Is it possible? Try it before you decide, because the answer is not the same.

Explain why the answer changed when the number 2 changed to 3.

WEEK 2 NUMBER STRUCTURE

Odd, Even and Impossible

Odd and even is not a label. It is a tool that rules things out.

IN CLASS THIS WEEK
Number patterns; place value (4.OA.5, 4.NBT.4)

NUMBER WARM-UP · 2 MINUTES
odd + odd = ?   odd + even = ?   even + even = ?

WORDS TO KEEP
parity whether a number is odd or even, thought of as a property that survives adding and subtracting

Fill this in from the warm-up. It is the whole toolkit for the week.

	+ even	+ odd
even		
odd		

1 Add a lot of them.

a. Add **four** even numbers. Odd or even? ________

b. Add **four** odd numbers. ________

c. Add **five** odd numbers. ________

d. Write the rule for adding **n** odd numbers.

____

2 Now rule things out.

Question	possible?	the reason
15 as the sum of 4 even numbers		
21 as the sum of 6 odd numbers		
21 as the sum of 5 odd numbers		
30 as the sum of 3 odd numbers		

3 One of them is possible.

Pick the row you marked possible and actually write the numbers out. Ruling something out is not the same as building it.

✎ ADD TO YOUR PROOF FILE

Proof File page 2: the parity table, and underneath it the sentence **a sum of even numbers is always even, so anything odd is out of reach.**

💬 TALK ABOUT IT

Why can a rule about odd and even settle a question about numbers you never wrote down?

Somebody says 15 is impossible because they tried lots of even numbers and none worked. Is that the same proof you gave?

STUCK? TRY THIS

Do not add the actual numbers. Ask only whether the total **has to be** odd or even, then compare that with the target.

★ CHALLENGE ZONE

Write the numbers 1 to 10 in a row. Put a + or a − in front of each one, however you like. Can the answer ever be exactly 0?

Work out the total of 1 to 10 first. Then think about what changing a + to a − does to that total.

WEEK 3 NUMBER STRUCTURE

Factor Rectangles

Every rectangle you can build from a number is a fact about that number.

IN CLASS THIS WEEK
Factors, multiples, prime and composite (4.OA.4)

NUMBER WARM-UP · 2 MINUTES
What times 6 makes 24? What times 8 makes 24?

WORDS TO KEEP

factor a number that divides another exactly, with nothing left over

factor pair two factors that multiply to make the number

Every rectangle that can be built from exactly **24** squares. You met this idea in Grade 3. This year it is a tool, not a picture.

24 × 1 12 × 2 8 × 3 6 × 4

1 Read the factors off the rectangles.

a. Write the four factor pairs of 24.

____

b. List every factor of 24, smallest first.

____

c. How many factors does 24 have? ________

2 Three more.

Number	factor pairs	all its factors	how many
36			
48			
13			

3 The square one.

a. Which of your numbers can be built as a **square**? ________

b. Count its factors. It is the only one with an **odd** count. Say why a square rectangle causes that.

____

✎ ADD TO YOUR PROOF FILE

Proof File page 3: the factor rectangles of 24, and the sentence explaining why a square number has an odd number of factors.

💬 TALK ABOUT IT

Why does every factor come in a pair? Where does the pairing break down?

Could a number have exactly three factors? Try to build one before you answer.

STUCK? TRY THIS

Divide by 1, then 2, then 3, and so on. Every exact division is a factor pair. Stop when the pairs start repeating themselves.

★ CHALLENGE ZONE

Which number below 50 has the most factors? Check every likely candidate rather than guessing.

Find every number below 50 with exactly three factors. They have something in common that next week explains.

WEEK 4 NUMBER STRUCTURE

Primes and Composites

Cross out every number that is a multiple of something. Look at what survives.

IN CLASS THIS WEEK
Prime and composite numbers (4.OA.4)

NUMBER WARM-UP · 2 MINUTES
Is 51 prime? Is 53?

WORDS TO KEEP

prime a number with exactly two factors, 1 and itself

composite a number with more than two factors

the sieve crossing out multiples until only primes are left

Circle **2**, then cross out every other multiple of 2. Circle **3**, cross out its other multiples. Then **5**. Then **7**. Stop there.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1 What survived.

a. How many primes are there below 100? ________

b. What is the largest one? ________

c. Why was it safe to stop at 7? The next prime is 11, and 11 × 11 = ________, which is bigger than 100.

2 Test three numbers properly.

Number	prime?	if not, a factor pair
91		
97		
87		

3 The awkward two.

a. Is 1 prime? Count its factors first. ________

b. Is 2 prime? It is the only even one, which makes people doubt it. ________

✎ **ADD TO YOUR PROOF FILE**

Proof File page 4: your list of primes below 100, and the reason the sieve could stop at 7. That reason is the useful part, not the list.

💬 **TALK ABOUT IT**

The sieve never asks whether a number is prime. It only crosses out. Why does that work?

Are the primes spread out evenly? Count how many are in each row of ten.

STUCK? TRY THIS

To test one number, try dividing by 2, 3, 5, 7, 11 and so on, and stop as soon as the divisor times itself passes your number.

★ **CHALLENGE ZONE**

Twin primes are two primes that differ by 2, like 11 and 13. Find every twin pair below 50.

Find the longest run of composite numbers below 100 with no prime in it.

Building From Primes

Break a number apart however you like. The bottom of the tree is always the same.

IN CLASS THIS WEEK
Factors and multiples; multi-digit multiplication (4.OA.4, 4.NBT.5)

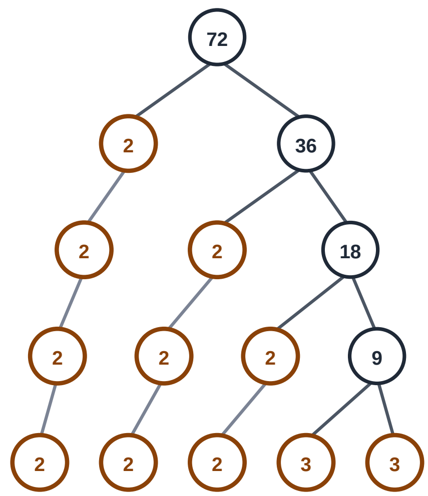
NUMBER WARM-UP · 2 MINUTES
What is 8 × 9? Now build 72 a different way.

WORDS TO KEEP

factor tree a picture of a number broken into factors, then those broken again, until only primes are left

prime factorization the list of primes at the bottom of the tree

A factor tree for **72**. The circled numbers are prime, so they stop.



1 Read the bottom row.

a. Write the primes at the bottom, in order.

____

b. Multiply them all together. What do you get? ________

2 Start it differently.

Build a new tree for 72, but split it as **8 × 9** at the top instead.

Did you get the same primes at the bottom? ________ In the same order? ________

3 Three more.

Number	its primes multiplied out
60	
100	
45	

✎ **ADD TO YOUR PROOF FILE**

Proof File page 5: both of your trees for 72 side by side, with the sentence **however you break it, the primes at the bottom are the same.**

💬 **TALK ABOUT IT**

Two people build different trees for the same number. Why can they never end up with different primes?

What is the factor tree of a prime number? Draw it.

STUCK? TRY THIS

It does not matter which factor pair you pick first. Split whatever is not yet prime, and keep going until every branch ends in a circled number.

★ **CHALLENGE ZONE**

A number has the prime factorization 2 × 2 × 2 × 3 × 5. What is the number?

Build the smallest number whose prime factorization uses exactly four primes, all different.

WEEK 6 NUMBER STRUCTURE

Multiples That Meet

Two rhythms started together. When do they line up again?

IN CLASS THIS WEEK
Factors and multiples (4.OA.4)

NUMBER WARM-UP · 2 MINUTES
Count by 4s to 24. Now count by 6s to 24.

WORDS TO KEEP

multiple what you get by counting up in steps of a number

common multiple a number that is on both lists

least common multiple the smallest number on both lists

Two lists. Circle the first number that appears on both.

count by 4	4	8	12	16	20	24	28	32	36
count by 6	6	12	18	24	30	36	42	48	54

1 Read it off.

a. The first number on both lists: ________

b. The next two after that: ________ and ________

c. What do you notice about all the numbers on both lists?

____

2 Do three more.

Pair	least common multiple
6 and 8	
5 and 7	
12 and 18	
4 and 12	

One pair behaved differently from the others. Which, and what is special about it?

3 Where the primes come in.

Write 12 and 18 as their primes. Now look at your answer of 36 as primes. Can you see where 36 came from?

✎ ADD TO YOUR PROOF FILE

Proof File page 6: your two lists with the meeting points circled, and what you noticed about every number that is on both.

💬 TALK ABOUT IT

Is there always a number on both lists? Say why you are sure.

When is the least common multiple just the two numbers multiplied together?

STUCK? TRY THIS

Write both lists out far enough to overshoot. The first shared number is the one you want, and every later one is a multiple of it.

★ CHALLENGE ZONE

Two buses leave the depot together at 9:00. One returns every 12 minutes, the other every 18. When are they next at the depot together?

Three buses now, every 4, 6 and 10 minutes. When do all three meet?

WEEK 7 NUMBER STRUCTURE

The Biggest Shared Factor

The other end of last week. Not where they meet, but what they share.

IN CLASS THIS WEEK
Factors and multiples (4.OA.4)

NUMBER WARM-UP · 2 MINUTES
Name every factor of 12. Now every factor of 18.

WORDS TO KEEP

common factor a number that divides both

greatest common factor the biggest number that divides both

Both factor lists, side by side. Circle every number that appears twice.

factors of 24	1	2	3	4	6	8	12	24	
factors of 36	1	2	3	4	6	9	12	18	36

1 Read it off.

a. Write every common factor.

____

b. The greatest common factor is ________

2 Three more pairs.

Pair	common factors	greatest
18 and 30		
15 and 28		
48 and 60		

One pair shares only the number 1. Which? ____________________

3 Use it for something.

You have **24** cookies and **36** brownies, and you want identical bags with nothing left over. What is the largest number of bags? _______ What goes in each bag?

____

✎ **ADD TO YOUR PROOF FILE**

Proof File page 7: the two factor lists with the shared ones circled, and the bag problem underneath, because that is what the greatest common factor is actually for.

💬 **TALK ABOUT IT**

Every pair of numbers shares at least one factor. Which one, and why always?

Last week you looked for where two lists meet. This week, where they overlap. Are those the same question?

STUCK? TRY THIS

List every factor of both numbers first, in full. The greatest common factor is the biggest number written on both lines, and it is often not the obvious one.

★ **CHALLENGE ZONE**

Two numbers whose greatest common factor is 1 are called **coprime**. Find three pairs of coprime numbers where neither number is prime.

A florist has 30 roses and 42 tulips and wants identical bunches. How many bunches, and what is in each?

WEEK 8 LOGIC & PROOF

Coloring Proves It

Color the board first. The coloring does the arguing.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

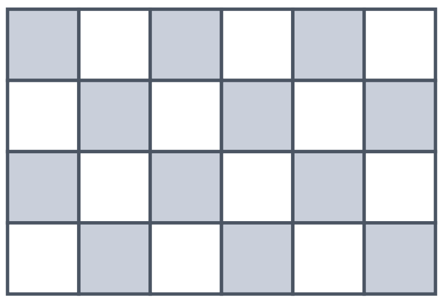
NUMBER WARM-UP · 2 MINUTES
A domino covers exactly two squares. On a colored board, how many of each color?

WORDS TO KEEP

domino a tile that covers exactly two squares side by side

tiling covering a board completely with no gaps and no overlaps

A colored board. Put one domino anywhere you like, going across or down.



1 The one fact that matters.

a. Your domino covers how many shaded squares? ________ How many white? ________

b. Try to place one that covers two shaded squares. Can you? ________ Why not?

____

c. So 12 dominoes would cover ________ shaded and ________ white.

2 Now use it.

For each board, count the colors first. Then say whether dominoes could possibly cover it.

Board	shaded	white	possible?
3 by 3			
6 by 4			
5 by 5			
6 by 4 with two white squares cut out			

3 Two different reasons.

Two of your boards are impossible. One is "impossible because of the **total** number of squares, the other because of the **colors**. Which is which?

✎ ADD TO YOUR PROOF FILE

Proof File page 8: a colored board with one domino drawn on it, and the sentence **every domino covers one of each color, always.** Everything next week rests on it.

💬 TALK ABOUT IT

The coloring is not part of the puzzle. Somebody added it. How can adding something prove anything?

If a board has equal colors, does that mean it definitely can be tiled?

STUCK? TRY THIS

Count the two colors before you try to place anything. If they are not equal, you can stop there, and if they are equal you still have to find an actual tiling.

★ CHALLENGE ZONE

A 5 by 5 board with the middle square removed leaves 24 squares. Count the colors. Does the coloring rule it out?

Then actually try to tile it. Compare what you found with what the coloring told you.

WEEK 9 LOGIC & PROOF

The Domino Problem

Sixty-two squares, thirty-one dominoes, and it cannot be done.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

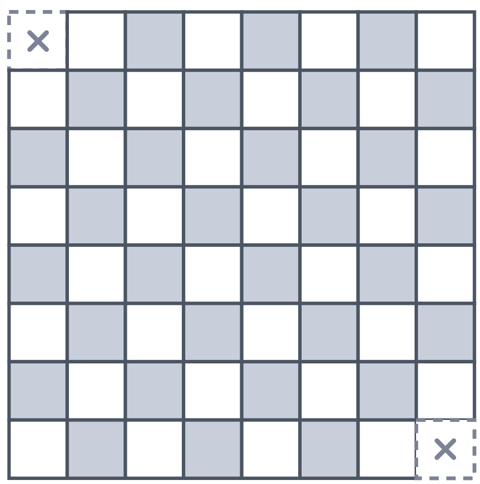
NUMBER WARM-UP · 2 MINUTES
How many squares on an 8 by 8 board? Take away 2.

WORDS TO KEEP

counterexample one case that shows a claim is false

proof an argument that covers every case at once, so no example is needed

An 8 by 8 board with two **opposite corners** cut out. Sixty-two squares are left, and you have thirty-one dominoes.



1 Finish the argument.

a. Both cut corners were the same color. Which? ____________________

b. So the board now has _______ shaded and _______ white.

c. Thirty-one dominoes would cover _______ of each color.

d. Write the contradiction in one sentence.

____

2 A different pair of corners.

a. Now cut out the **top left** and the **top right** corners instead. What colors are they?

____

b. Does the coloring rule this one out? _______ Is it actually possible? _______

3 State the rule.

Cut out any two squares you like. When is the board definitely impossible?

____

✎ ADD TO YOUR PROOF FILE

Proof File page 9. This is the best page in the book so far, so lay it out properly: the claim, the coloring, the two counts, and the contradiction. Four lines.

💬 TALK ABOUT IT

Nobody has ever tried all the ways to place thirty-one dominoes. Why do we not need to?

The second board is not ruled out by the coloring. Does that make it possible? What would you still have to do?

STUCK? TRY THIS

Count the shaded squares and the white squares separately, and then count what thirty-one dominoes must cover. The two numbers cannot both be right.

★ CHALLENGE ZONE

Cut out one shaded square and one white square, anywhere on the board, not just corners. Is it always still possible? Try several.

Now cut out four squares, two of each color, so that it becomes impossible anyway. The coloring is not the only thing that can go wrong.

Degree Tells You

Add up every degree in any drawing at all. The answer is never odd.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

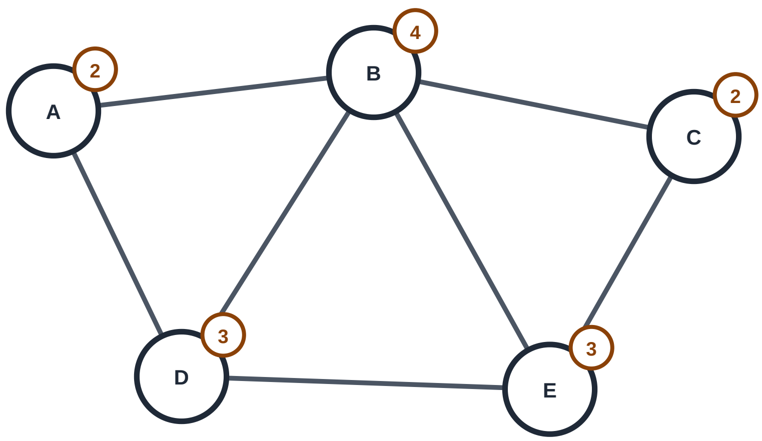
NUMBER WARM-UP · 2 MINUTES
A line has two ends. If you draw 5 lines, how many ends is that?

WORDS TO KEEP

degree how many lines meet at one dot

theorem a claim that has been proved, so it can be used without checking again

Every line has **two** ends, and each end lands on a dot. That single fact is the whole week.



1 Count it twice.

a. Add up all five degrees. ________

b. Count the lines. ________ Double it. ________

c. Write the theorem: the degrees always add up to ______________________

2 The second theorem, which is the useful one.

a. In the picture, how many dots have an **odd** degree? ________

b. Draw three graphs of your own and count the odd dots in each.

c. The number of odd dots is always ______________________

d. Why? The total is even. Every even degree adds an even amount. So the odd ones must come in ______________________

3 Now rule things out.

Claim	impossible?	the reason
7 houses, each joined to exactly 3 others		
6 houses, each joined to exactly 3 others		
9 people, each shook exactly 5 hands		

✎ ADD TO YOUR PROOF FILE

Proof File page 10: both theorems, each with the one-line reason. You will use the second one for the rest of the year and again in Grade 6.

TALK ABOUT IT

Why is it impossible for exactly three dots to have odd degree?

Does the theorem tell you a graph exists when the count comes out even?

STUCK? TRY THIS

Do not draw the graph first. Multiply the number of dots by the degree each one is supposed to have. If that total is odd, stop, because no drawing can have an odd total.

★ CHALLENGE ZONE

Can you draw a graph with 5 dots where four of them have degree 3 and one has degree 2?

In any room of people, prove that the number who shook an odd number of hands is even. Grade 3 asked you to notice this. Now write the proof.

WEEK 11 DOTS & LINES

Trace It or Prove You Cannot

The old bridge puzzle, settled in one line, without walking anywhere.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

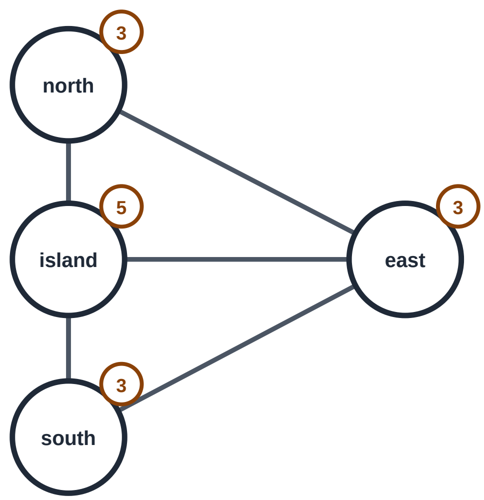
NUMBER WARM-UP · 2 MINUTES
How many odd numbers are in this list: 5, 3, 3, 3?

WORDS TO KEEP

trace draw the whole figure without lifting your pencil and without going over any line twice

the tracing rule a connected figure can be traced only if it has **0** or **2** odd dots. No other number works

Four pieces of land in the old city of Königsberg, joined by seven bridges. People tried for years to walk over every bridge exactly once.



1 Settle it in three lines.

a. Write the four degrees. ______________________

b. How many of them are odd? ________

c. The rule allows 0 or 2. So the walk is ______________________

2 Why the rule is what it is.

a. Think about a dot in the **middle** of your walk. You arrive on one line and leave on another, so its lines get used ________ at a time.

b. So a middle dot must have an ____________________ degree. Only the start and the finish can be odd, and that is why the answer is 0 or 2.

3 Four figures.

Figure	odd dots	traceable?
a square		
a square with one diagonal		
a square with both diagonals		
an envelope: a square, both diagonals and a roof		

✎ ADD TO YOUR PROOF FILE

Proof File page 11: the bridges, their four degrees, and the one sentence that ends the puzzle. Note that nobody has to walk anywhere.

💬 TALK ABOUT IT

People tried this walk for years. What did the rule give them that trying never could?

If a figure has 2 odd dots, where must you start?

STUCK? TRY THIS

Count the lines meeting at each dot and write the number beside it. Then count how many of those numbers are odd. That count is the entire answer.

★ CHALLENGE ZONE

Draw a figure with exactly 2 odd dots, then trace it. Where did you have to start and finish?

Add one more bridge to Königsberg so that the walk becomes possible. There is more than one answer, and each one changes two degrees at once.

Impossible Fair

Six claims. Prove each one, or knock it down. You may not shrug.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

NUMBER WARM-UP · 2 MINUTES
No warm-up. Read all six first and start with whichever looks easiest.

WORDS TO KEEP

prove give a reason that covers every case

disprove give one single example where the claim fails. That is enough

Three of these are true and three are false. For a true one you must give the **reason**. For a false one, one example is enough to sink it.

The claims

1. The sum of three odd numbers is always odd.
2. Every number with an odd number of factors is a square.
3. Seven people can each shake hands with exactly three others.
4. Every prime number is odd.
5. A 4 by 6 board with one square removed can be tiled with dominoes.
6. If a number is divisible by 2 and by 3, it is divisible by 6.

1 Your answers

#	true or false	proof, or the example that kills it
1		
2		
3		
4		
5		
6		

2 Which tool did each one need?

Write the tool beside each number: parity, factors, degree, coloring, primes.

____

____

✎ ADD TO YOUR PROOF FILE

Proof File page 12: the three true claims with their reasons. Leave the false ones out of the file, but write the killing example beside each on a separate sheet.

💬 TALK ABOUT IT

Disproving takes one example and proving takes an argument. Why is that fair?

Claim 4 is knocked down by a single number. Does that make it a bad claim, or a useful one?

STUCK? TRY THIS

For each claim, ask first whether you believe it. If you do, hunt for the reason. If you do not, hunt for one example, and check it really does break the claim.

★ CHALLENGE ZONE

Write a seventh claim of your own that is false, but only just: it should take somebody a while to find the example. Swap with somebody else.

WEEK 13 LOGIC & PROOF

Case Files

Everything from the first twelve weeks, in one folder.

IN CLASS THIS WEEK
Review and reasoning

NUMBER WARM-UP · 2 MINUTES
No warm-up this week.

Five cases. Each one needs the answer **and** the tool.

1 The locked cups

Nine cups upside down. You may turn over exactly two at a time. Can you get them all up? Which invariant decides it?

2 The shared factor

What is the greatest common factor of 42 and 70? And their least common multiple?

3 The village

Eleven houses, each to be joined to exactly five others. Possible?

4 The cut board

A 6 by 6 board with the top left and bottom right squares removed. Can 17 dominoes cover it?

5 The tree

Write 84 as a product of primes. How many factors does 84 have altogether?

Which tool?

Beside each case number write its tool: parity invariant ________, common factors ________, degree ________, coloring ________, prime factorization ________.

✎ ADD TO YOUR PROOF FILE

Add a Proof File page called **My tools so far**: parity, factors and primes, degree, and coloring. One line each, in the words you would use out loud.

TALK ABOUT IT

Four of these five are settled without trying a single arrangement. Which one is not?

Which tool has been the most useful so far? Which was hardest to trust?

STUCK? TRY THIS

Before reaching for a method, ask what is being **counted** and whether that count can change. Nearly every case this trimester turns on a count that cannot.

★ CHALLENGE ZONE

Invent a sixth case that looks like it needs coloring but actually needs parity. Write an answer key on a separate sheet.

WEEK 14 NUMBER STRUCTURE

Divisible by 2, 5 and 10

Why does only the last digit matter? There is a reason, and it is short.

IN CLASS THIS WEEK
Place value to one million (4.NBT.1, 4.NBT.2)

NUMBER WARM-UP · 2 MINUTES
Is 3,470 divisible by 10? By 5? By 2?

WORDS TO KEEP

divisible by divides exactly, with remainder 0

the reason every place above the ones is worth a multiple of 10, and 10 is 2 × 5

Break **3,470** apart by place value. Look at what each piece is divisible by.

Piece	value	divisible by 2?	by 5?	by 10?
3 thousands	3000			
4 hundreds	400			
7 tens	70			
0 ones	0			

1 So the rule falls out.

a. Every piece except the **ones** is divisible by 10, and therefore by ________ and by ________.

b. So the whole number is divisible by 2 exactly when the ______________________ is.

2 Test them.

Number	by 2	by 5	by 10	by 4
3,470				
8,216				
9,005				
12,348				

3 Why 4 needs two digits.

a. Is 100 divisible by 4? ________ Is 1,000? ________

b. So for 4, everything above the **hundreds** takes care of itself, and you only need the last ________ digits.

✎ **ADD TO YOUR PROOF FILE**

Proof File page 13: the three rules for 2, 5 and 10, each with its reason in one line. A rule without its reason does not go in the file.

💬 **TALK ABOUT IT**

Why is there no last-digit rule for 3? Test 12 and 21 and 13 to see the trouble.

What would the rule for 8 be, and how many digits would you need?

STUCK? TRY THIS

Split the number into thousands, hundreds, tens and ones, and check each piece separately. Every piece that is already divisible can be ignored.

★ **CHALLENGE ZONE**

Work out the rule for 8 and test it on 1,048 and on 3,036.

A number is divisible by 20. What must be true about its last two digits?

Divisible by 3 and 9

Add the digits. It is not a trick, and here is why it works.

IN CLASS THIS WEEK
Place value and multi-digit arithmetic (4.NBT.1, 4.NBT.4)

NUMBER WARM-UP · 2 MINUTES
Add the digits of 4,275. Is that total divisible by 9?

WORDS TO KEEP

digit sum add all the digits of a number

the reason 10 is 9 + 1, and 100 is 99 + 1, and 1,000 is 999 + 1. The 9s and 99s always divide by 9, so only the leftover 1s matter

Take **4,275** apart, and split each place into a run of nines plus a leftover.

Place	written out	the nines part	the leftover
4 thousands	4 × 999 + 4	4 × 999	4
2 hundreds	2 × 99 + 2	2 × 99	2
7 tens	7 × 9 + 7	7 × 9	7
5 ones	5	0	5

1 Read the last column.

a. The leftovers are 4, 2, 7, 5. What are those? ______________________

b. Every nines part divides by 9. So 4,275 is divisible by 9 exactly when ______________________ is.

2 Test six numbers.

Number	digit sum	by 3?	by 9?
4,275			
1,236			
8,001			
5,555			

A number divisible by 9 is always divisible by 3. Is the reverse true? ________

3 Put two rules together.

A number is divisible by 6 exactly when it is divisible by ________ and by ________. Test 1,236 and 8,001 for divisibility by 6.

____

✎ ADD TO YOUR PROOF FILE

Proof File page 14: the digit-sum rule with the 9 + 1 reason. This is the page next week depends on, so write the reason out in full.

TALK ABOUT IT

Why does the same digit sum work for both 3 and 9?

If you scramble the digits of a number, does it stay divisible by 9? Say why.

STUCK? TRY THIS

You may add the digits again if the total is still big. 4 + 2 + 7 + 5 = 18, and 1 + 8 = 9, which is easier to judge.

★ CHALLENGE ZONE

Is 123,456,789 divisible by 9? Do it in your head.

Find the smallest number greater than 1,000 that is divisible by both 8 and 9.

WEEK 16 NUMBER STRUCTURE

Casting Out Nines

A real check on multiplication, from the rule you proved last week.

IN CLASS THIS WEEK
Multi-digit multiplication (4.NBT.5)

NUMBER WARM-UP · 2 MINUTES
Digit sum of 27? Of 34? Keep adding until one digit is left.

WORDS TO KEEP

casting out nines replacing a number by its digit sum, over and over, until one digit is left

a check a test that catches wrong answers. It does not promise a right one

Somebody worked out **27 × 34 = 918**. Check it without multiplying again.

	the number	cast down to one digit
first factor	27	
second factor	34	
multiply those two		
their answer	918	

1 Compare the last two rows.

a. Do they match? ________

b. If they had **not** matched, what would you know for certain?

____

2 Three answers. One is wrong.

Multiplication	left side cast	answer cast	wrong?
43 × 26 = 1,118			
58 × 37 = 2,156			
64 × 45 = 2,880			

Which one is definitely wrong? ________ Work out the right answer. ________

3 What the check cannot do.

Somebody wrote 918 as 981 by mistake. Cast both down. ________ and ________. Does the check spot the error? ________ Say why not.

____

✎ ADD TO YOUR PROOF FILE

Proof File page 15: the check written as three steps, and underneath it, honestly, the kind of mistake it will never catch.

💬 TALK ABOUT IT

A check that passes does not mean the answer is right. Is a check like that worth having?

Roughly how many wrong answers would slip past it? Think about how many possible single digits there are.

STUCK? TRY THIS

Cast each factor down to one digit, multiply those two small numbers, then cast that down too. Compare it with the answer cast down. Two different numbers means a mistake.

★ CHALLENGE ZONE

Make up a multiplication, get it deliberately wrong, and check whether casting out nines catches you.

Does the same check work for addition? Try it on three sums and say what you find.

WEEK 17 NUMBER STRUCTURE

Wrapping Around, Again

Casting out nines was not a trick. It was arithmetic on a wheel.

IN CLASS THIS WEEK
Multi-digit arithmetic and patterns (4.NBT.4, 4.OA.5)

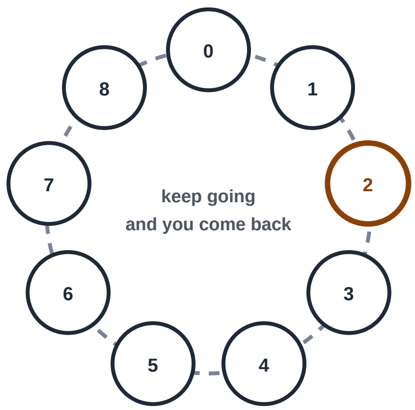
NUMBER WARM-UP · 2 MINUTES
What is the remainder when 20 is divided by 9? When 38 is?

WORDS TO KEEP

mod short for the remainder after dividing. 38 mod 9 is 2

the wheel the numbers 0 to 8 in a ring. Counting past 8 lands back on 0

The nine remainders, in a ring. Every number in the world sits on one of these nine places.



1 Put numbers on the wheel.

Number	÷ 9 leaves	Number	÷ 5 leaves
38		38	
100		100	
918		918	

Compare your 918 answer with its digit sum from last week. ______________________

2 Why the two weeks are the same week.

Casting out nines replaces a number by its digit sum. The wheel replaces it by its remainder after 9. Write one sentence saying why those give the same place on the wheel.

____

____

3 Powers on a wheel.

Work out the **last digit** of each, which is the same as working mod 10.

	3	3 × 3	× 3	× 3	× 3	× 3
last digit	3	9				

How long before the pattern repeats? ________

✎ ADD TO YOUR PROOF FILE

Proof File page 16: the wheel of nine, with 918 marked on it and the digit-sum connection written underneath.

💬 TALK ABOUT IT

Why does every number land somewhere on the wheel, however big it is?

The last digit of 3, 9, 27, 81 repeats. What is the last digit of 3 multiplied by itself twenty times? You do not need to work it out.

STUCK? TRY THIS

Divide and keep only the remainder. Everything else about the number can be thrown away, which is exactly what makes the wheel useful.

★ CHALLENGE ZONE

What is the last digit of 7 multiplied by itself 30 times? Find the cycle first.

Which numbers land on 0 on the nine-wheel? Describe them without listing them.

WEEK 18 NUMBER STRUCTURE

The Day of the Week

A whole year on a seven-wheel, and no counting on fingers.

IN CLASS THIS WEEK
Multi-step problems; measurement (4.OA.3, 4.MD.2)

NUMBER WARM-UP · 2 MINUTES
What is 59 ÷ 7? What is the remainder?

WORDS TO KEEP

offset how many days a month starts after January 1st

the seven-wheel only the remainder after dividing by 7 changes the day

In a year that is **not** a leap year, **January 1st is a Monday**. The offset is how many days later each month begins.

Month	offset	÷ 7 leaves	starts on
January	0		
February	31		
March	59		
April	90		
May	120		
June	151		

1 Do March first.

a. March 1st is 59 days after January 1st. 59 ÷ 7 = ________ remainder ________

b. Move that many days on from Monday. ________________________

2 Finish the second half of the year.

Month	offset	÷ 7 leaves	starts on
July	181		
August	212		
September	243		
October	273		
November	304		
December	334		

3 Look down the last column.

a. Which month starts on the same day as **January**? ______________________

b. Find two more months that share a starting day with each other.

____

✎ **ADD TO YOUR PROOF FILE**

Proof File page 17: the twelve offsets with their remainders after 7. Any year that starts on a Monday and is not a leap year works exactly the same way.

💬 **TALK ABOUT IT**

The offsets grow to 334 and the answers only ever take 7 values. Why?

In a leap year, which months would shift and which would not?

STUCK? TRY THIS

Divide the offset by 7 and throw away the whole weeks. Only the remainder moves you round from Monday.

★ **CHALLENGE ZONE**

February 1st and March 1st start on the same day in a normal year. Why does that happen, and what does it tell you about the length of February?

In a leap year, does that pair still match? Work it out rather than guessing.

WEEK 19 SHAPE & SPACE

Lines of Symmetry

Fold it. If the two halves land on each other, you found a line.

IN CLASS THIS WEEK
Lines of symmetry in two-dimensional figures (4.G.3)

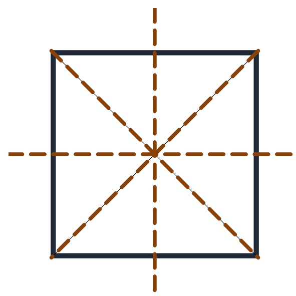
NUMBER WARM-UP · 2 MINUTES
How many ways can you fold a square in half so the halves match?

WORDS TO KEEP

line of symmetry a fold line where one half lands exactly on the other

regular a shape with all sides equal and all corners equal

A square has four fold lines. Two go through the middles of the sides and two go corner to corner.



1 Check each one.

a. Fold along the up-and-down line. Does every corner land on a corner? ________

b. Fold corner to corner. Same question. ________

c. Try a fifth line, anywhere else. What goes wrong?

____

2 Count the lines for each shape.

Shape	lines of symmetry
square	4
rectangle that is not a square	
equilateral triangle	
isosceles triangle	
regular hexagon	
regular pentagon	
parallelogram that is not a rectangle	
scalene triangle	

3 Two of them got zero.

Both zeroes surprise people, because the shapes still look balanced. Draw the parallelogram and show a fold that **nearly** works, then say exactly what goes wrong.

✎ **ADD TO YOUR PROOF FILE**

Proof File page 18: the eight shapes with their line counts. Circle the two that have none, because next week is about them.

💬 **TALK ABOUT IT**

A parallelogram looks symmetrical and has no fold line. What is your eye seeing?

Does a shape with more sides always have more lines of symmetry?

STUCK? TRY THIS

Trace the shape, cut it out and actually fold it. A line that only nearly works does not count, and folding is the fastest way to find that out.

★ **CHALLENGE ZONE**

A regular polygon with 7 sides. How many lines of symmetry? Now one with 12 sides. Write the rule.

Draw a shape with exactly two lines of symmetry that is **not** a rectangle.

WEEK 20 SHAPE & SPACE

Turning a Shape

Fold lines are not the only kind of symmetry. Some shapes only turn.

IN CLASS THIS WEEK
Lines of symmetry; angle measure (4.G.3, 4.MD.5)

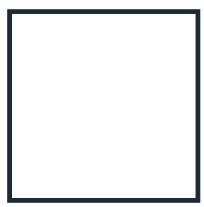
NUMBER WARM-UP · 2 MINUTES
A quarter turn is how many degrees? A half turn?

WORDS TO KEEP

turn symmetry turning a shape less than all the way round and having it land on itself

order how many turns, including the full turn back to the start, land it on itself

Turn a square a quarter turn. It lands on itself. Keep going and count how many landings you get before you are back where you started.



1 Turn it round.

Turn	0	quarter	half	three quarters	full
lands on itself?	yes				

So the square has turn symmetry of order ________

2 Fill both columns.

Shape	fold lines	turn order
square	4	4
rectangle, not a square		
equilateral triangle		
regular hexagon		
parallelogram, not a rectangle		
isosceles triangle		

3 The interesting row.

a. Which shape has turn symmetry but **no** fold lines at all? ____________________

b. So the two kinds of symmetry are not the same thing. Is there a shape with fold lines but no turn symmetry beyond the full turn? ____________________

✎ **ADD TO YOUR PROOF FILE**

Proof File page 19: the two-column table. The parallelogram row is the point of the page, so mark it.

💬 **TALK ABOUT IT**

Why does every shape have turn order at least 1?

A shape has 5 fold lines. What do you expect its turn order to be? Check with the pentagon.

STUCK? TRY THIS

Trace the shape on tracing paper, put a pin through the middle and turn the tracing. Count how many times the tracing fits the drawing underneath.

★ **CHALLENGE ZONE**

Draw a shape with turn order 3 and no fold lines. A pinwheel is a good starting point.

For a regular polygon, how are the number of fold lines and the turn order related? Say why.

The Eight Ways to Move a Square

Every move that leaves a square looking the same. There are exactly eight.

IN CLASS THIS WEEK
Angle measure and reasoning (4.MD.5, 4.OA.3)

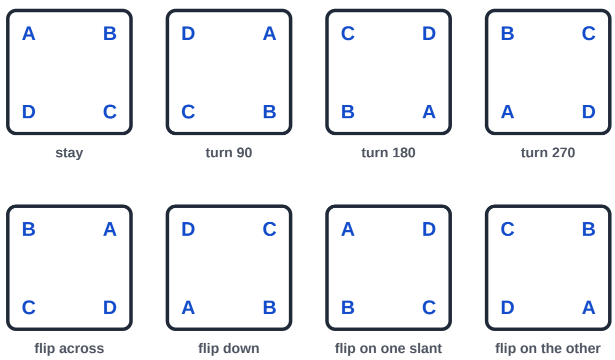
NUMBER WARM-UP · 2 MINUTES
A quarter turn and then another quarter turn is the same as what?

WORDS TO KEEP

a move here it means turning or flipping a square so it lands exactly on itself

composing doing one move and then another. The result is always one of the eight

The corners are labeled A, B, C, D so you can see where each one goes. Four turns and four flips, and that is all of them.



1 Check the count.

a. How many turns? ________ How many flips? ________ Total: ________

b. Is there a ninth? Try to invent one, then find it in the picture.

____

2 Do two in a row.

Follow the corner letters through both moves, then find the single move that would have done the same thing.

first move	then this move	same as
turn 90	turn 90	
turn 90	turn 270	
flip across	flip down	
flip across	flip across	

3 Undo.

a. Which moves get you back to the start when you do them **twice**? List them.

____

b. Which move undoes **turn 90**? ____________________

✎ **ADD TO YOUR PROOF FILE**

Proof File page 20: the eight moves listed, and the fact that doing any two of them in a row always lands you on another one of the eight. That closed-ness is the surprise.

💬 **TALK ABOUT IT**

Doing two moves always gives a move already on the list. Why could it not give a ninth?

Which single move is the odd one out, in the sense that it changes nothing?

STUCK? TRY THIS

Track one corner, say A, all the way through. Where A ends up plus whether the square got flipped over is enough to name the move.

★ **CHALLENGE ZONE**

How many moves does an **equilateral triangle** have? Count turns and flips.

And a rectangle that is not a square? The answer is smaller than eight, and working out which moves are lost is the puzzle.

WEEK 22 SHAPE & SPACE

Angles That Must Add Up

Tear the three corners off any triangle at all. They make a straight line.

IN CLASS THIS WEEK
Angle measure and adding angles (4.MD.5, 4.MD.7)

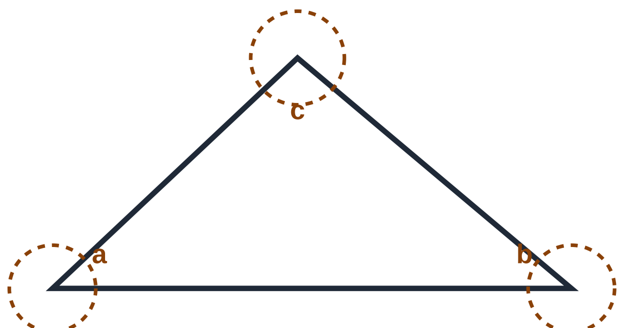
NUMBER WARM-UP · 2 MINUTES
How many degrees in a straight line? In a quarter turn?

WORDS TO KEEP

degree the unit for measuring turn. A full turn is 360 degrees

straight angle half a turn, which is 180 degrees

Draw any triangle you like on scrap paper. Tear off all three corners and put them together, points touching.



a + b + c = ?

1 What you get.

a. The three corners together make a ____________________

b. So a + b + c = _______ degrees.

c. Do it again with a completely different triangle. Same answer? _______

2 Find the missing angle.

a	b	c	possible?
60	70		
90	45		
30	30		
90	95		

One row is impossible. Which, and why?

____

3 More than three sides.

a. Draw a four-sided shape and cut it into **two** triangles with one straight line. So its angles add to ________

b. A five-sided shape splits into ________ triangles, so its angles add to ________

✎ **ADD TO YOUR PROOF FILE**

Proof File page 21: the torn-corner picture, the 180, and the splitting-into-triangles trick that turns it into a rule for any shape.

💬 **TALK ABOUT IT**

The tearing works for every triangle anybody has ever drawn. Is that a proof?

Can a triangle have two right angles? Use the rule rather than trying to draw one.

STUCK? TRY THIS

For a shape with more sides, draw all the lines from one corner. Count the triangles you made, then multiply by 180.

★ **CHALLENGE ZONE**

Work out the angle total for a six-sided shape and a ten-sided shape.

Write the rule for a shape with **n** sides. You will need to say how many triangles it splits into first.

WEEK 23 SHAPE & SPACE

Tiling the Floor

Only three regular shapes tile a floor. The reason is one division.

IN CLASS THIS WEEK
Angle measure (4.MD.5, 4.MD.7)

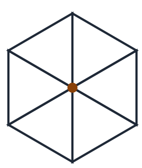
NUMBER WARM-UP · 2 MINUTES
What is 360 ÷ 60? What is 360 ÷ 90? What is 360 ÷ 108?

WORDS TO KEEP

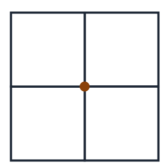
tile cover a flat floor with copies of one shape, no gaps and no overlaps

corner angle the angle inside a shape at one corner. Copies meeting at a point must have corner angles adding to exactly 360

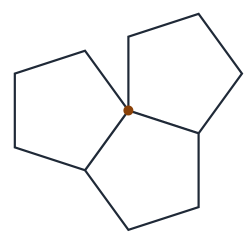
Put copies of one shape around a single point. They fit only if the corner angles add to exactly **360**.



each corner is 60 degrees
6 of them fit, they close up exactly



each corner is 90 degrees
4 of them fit, they close up exactly



each corner is 108 degrees
3 of them fit, 36 degrees left over

1 Read the three pictures.

Triangles fit ________ at a point, squares fit ________, and pentagons leave ________ degrees over.

2 Do the division for each shape.

Regular shape	corner angle	360 ÷ that	tiles?
triangle	60		
square	90		
pentagon	108		
hexagon	120		
octagon	135		

3 Why it stops.

a. Which three shapes tile? ______________________

b. A shape with more than six sides has a corner angle bigger than 120 and smaller than 180. So 360 divided by it lands between ________ and ________, and can never be a whole number. That is the whole proof.

✎ **ADD TO YOUR PROOF FILE**

Proof File page 22: the three shapes that tile, the division that decides it, and the argument that nothing with seven or more sides can ever work.

💬 **TALK ABOUT IT**

Bathroom floors are usually square tiles and honeycombs are hexagons. Does the rule explain both?

Pentagons leave 36 degrees over. Could you fill that gap with a different shape?

STUCK? TRY THIS

Work out the corner angle first, then divide 360 by it. Only a whole-number answer lets the copies close up.

★ **CHALLENGE ZONE**

Two regular octagons meeting at a point use 135 + 135 degrees. What angle is left, and which regular shape fits it exactly? Real bathroom floors are tiled this way.

Any triangle at all, even a lopsided one, tiles a floor. Try it with six copies of a scalene triangle and say why it works when regular pentagons do not.

WEEK 24 SHAPE & SPACE

Impossible Tilings

Two boards, one corner different, and the answers are not the same.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

NUMBER WARM-UP · 2 MINUTES
Is 64 divisible by 3? Is 63?

WORDS TO KEEP

counting argument the number of squares must divide by the size of the tile, or stop right there

coloring argument when counting is not enough, color the board so that every tile must cover one of each color

The first question is always the easy one: does the number of squares even divide by the size of the tile?

Board	squares	tile	divides?	possible?
5 by 5	25	domino (2)		
4 by 4	16	L-tromino (3)		
8 by 8	64	1 by 3 (3)		
6 by 4	24	1 by 3 (3)		

1 When counting is not enough.

The last row divides exactly. Does that mean it can be done? ________ What would you still have to do?

____

2 Color it in threes.

Number every square by **row + column**, then color it by the remainder after 3. A 1 by 3 tile, lying either way, always covers one of each color.

a. On a full 8 by 8 board the three color counts are **21**, **22** and **21**. Are they equal? ________

b. That already rules out the whole board, but so did counting. Now take one square away, leaving 63.

3 The two corners.

a. Remove the **top left** corner. It is a color there are 21 of. The counts become 20, 22, 21. Equal? ________ So it is ______________________

b. Remove the **top right** corner instead. It is the color there are 22 of, so the counts become 21, 21, 21. Does the coloring rule this one out? ________

✎ ADD TO YOUR PROOF FILE

Proof File page 23: the two corners side by side. Same board, same tiles, and one is settled while the other is left open. Write down which is which and why.

💬 TALK ABOUT IT

The coloring rules out one corner and says nothing about the other. Is the second one therefore possible?

Why does a 1 by 3 tile always cover one of each color, whichever way round it lies?

STUCK? TRY THIS

Do the counting test first, because it is quick and it settles most boards. Only reach for a coloring when the counting comes out even and you still cannot do it.

★ CHALLENGE ZONE

Take away one square from a 5 by 5 board so that dominoes could cover the rest. Which squares work?

A 9 by 9 board has 81 squares and its three color counts are all 27. What does that tell you, and what does it not tell you?

WEEK 25 LOGIC & PROOF

Puzzle Fair

A little of everything from Weeks 14 to 24.

IN CLASS THIS WEEK
Review and reasoning

NUMBER WARM-UP · 2 MINUTES
No warm-up. Start wherever you like.

Six puzzles. Answer plus tool, every time.

1 The long division

Is 71,325 divisible by 3? By 9? By 5? Do all three without dividing.

2 The bad answer

Somebody says 46 × 27 = 1,232. Check it by casting out nines.

3 The folded shape

A shape has 6 fold lines. What is its turn order, and what shape is it?

4 The floor

Could a floor be tiled with regular nine-sided shapes? Find the corner angle first.

5 The calendar

In a year starting on Monday that is not a leap year, what day is December 1st? Its offset is 334.

6 The cut board

A 7 by 7 board has one corner removed. Can dominoes cover the remaining 48 squares? Count the colors.

✎ ADD TO YOUR PROOF FILE

Add a Proof File page called **My tools, weeks 14 to 24**: divisibility with reasons, casting out nines, the seven-wheel, symmetry, corner angles, and coloring.

💬 TALK ABOUT IT

Puzzle 6 needs both a count and a coloring. Which one settles it?

Which puzzle could you do entirely in your head? What made it easy?

STUCK? TRY THIS

Almost everything this trimester is either a remainder or a count of two colors. Work out which one a puzzle is asking about before you start.

★ CHALLENGE ZONE

Write a seventh puzzle that looks like a divisibility question but is really about symmetry. Include an answer key on a separate sheet.

Choosing a Team

Line them up, then divide out the orderings you did not want.

IN CLASS THIS WEEK
Multi-step problems (4.OA.3)

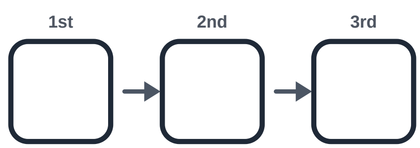
NUMBER WARM-UP · 2 MINUTES
What is 5 × 4 × 3? What is 3 × 2 × 1?

WORDS TO KEEP

arrangement an order. Ana then Ben is different from Ben then Ana

combination a choice where the order does not matter, so those two are one team

Five children, and **three** of them go on the trip. Count it in two steps: first count the line-ups, then divide out the ones that are the same team.



first pick, then second, then third

1 Step one, then step two.

a. How many for the first pick? ________ Then ________, then ________. Multiply: ________

b. Take one team, say Ana, Ben, Cal. How many different orders could those same three be picked in? ________

c. So divide. Number of teams: ________

2 Four more.

Choose	line-ups	orders of one team	teams
2 from 6	6 × 5 = 30	2	
3 from 6			
4 from 6			
3 from 7			

3 Two of your answers match.

Choosing 2 from 6 and choosing 4 from 6 give the same number. That is not a coincidence. Say why, thinking about the children you **leave behind**.

✎ ADD TO YOUR PROOF FILE

Proof File page 24: the two-step method, and the leaving-behind reason for why choosing 2 and choosing 4 from the same six come out equal.

TALK ABOUT IT

Why do you divide rather than subtract?

Choosing 0 children from 6: how many ways? The answer is not 0.

STUCK? TRY THIS

Count the line-ups first and do not worry that you are over-counting. Then work out exactly how many times each team got counted, and divide by that.

★ CHALLENGE ZONE

How many ways to choose 3 from 8? And 5 from 8? Check that the leaving-behind rule predicts the second from the first.

A pizza shop has 7 toppings and you pick 3. How many pizzas?

The Triangle Grows

Every number is the two above it added together, and that rule builds the whole thing.

IN CLASS THIS WEEK
Number and shape patterns (4.OA.5)

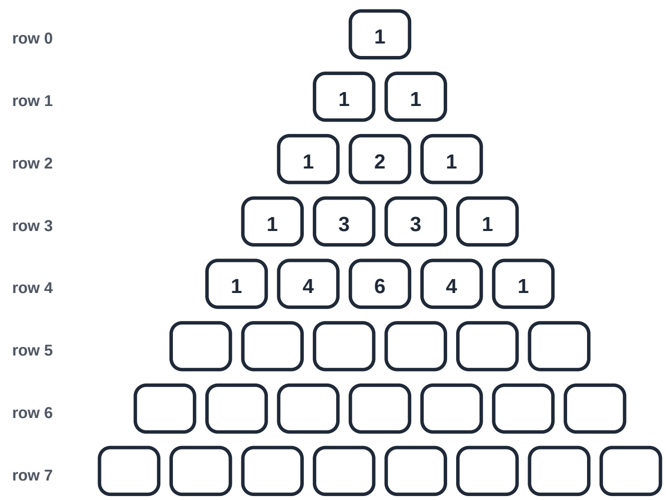
NUMBER WARM-UP · 2 MINUTES
1 + 4 = ?   4 + 6 = ?   10 + 10 = ?

WORDS TO KEEP

recurrence a rule that builds the next line out of the line before it

row the triangle's rows are numbered from 0, so row 4 is the fifth line down

You met this triangle in Grade 3. This year you build it from its rule and then use it. Rows 0 to 4 are filled in.



1 Build three more rows.

a. Row 5: 1, ________, ________, ________, ________, 1

b. Fill in rows 6 and 7 on the picture.

2 Two patterns to find.

a. Add up each row. Write the totals for rows 0 to 7.

____

b. Each total is the one before it ____________________

c. Read down the third number in each row: 1, 3, 6, 10, 15, 21. Where did you meet those in Grade 3? ____________________

3 Why the edges are always 1.

A number on the edge has only **one** number above it, not two. Explain what the missing one counts as.

____

✎ ADD TO YOUR PROOF FILE

Proof File page 25: seven rows of the triangle with the row totals down one side. Next week connects this page to last week's teams.

💬 TALK ABOUT IT

The rule only ever adds. So why do the numbers grow so fast?

Row 7 adds up to 128. Predict the total of row 10 without building it.

STUCK? TRY THIS

Work one row at a time, left to right, and write each number under the gap between the two you added. The edges are always 1.

★ CHALLENGE ZONE

Build rows 8, 9 and 10.

The biggest number in row 10 is in the middle. What is it, and how does it compare with the row total?

The Triangle and the Teams

Last week's triangle and the week before's teams are the same numbers.

IN CLASS THIS WEEK
Number and shape patterns (4.OA.5)

NUMBER WARM-UP · 2 MINUTES
How many ways to choose 2 children from 4? Look at row 4 of the triangle.

WORDS TO KEEP

row n the row of the triangle that starts 1, then n

choosing k from n the number of teams of k you can pick out of n people

Row 6 of the triangle is **1, 6, 15, 20, 15, 6, 1**. Underneath it, work out the teams you can choose from six children.

Choose from 6	0	1	2	3	4	5	6
how many teams							
row 6 says	1	6	15	20	15	6	1

1 They match.

a. Choosing 2 from 6 gave you 15 last week. Where is 15 in row 6? ______________________

b. Choosing **0** from 6 is 1 way, which is doing nothing. Does that fit the row? ________

2 Read answers straight off the triangle.

Question	row	position	answer
choose 3 from 6	6	3	
choose 2 from 7			
choose 3 from 7			
choose 4 from 8			

3 What the row total counts.

a. Row 6 adds up to ________

b. Each child either comes or stays home, so there are 2 choices each, six times over: 2 × 2 × 2 × 2 × 2 × 2 = ________. Same number. Say why.

____

✎ **ADD TO YOUR PROOF FILE**

Proof File page 26: row 6 with the team counts written underneath it, and the come-or-stay-home argument for the row total.

💬 **TALK ABOUT IT**

Why does the triangle answer a question about teams when it was built by adding?

Row 6 reads the same forwards and backwards. Which fact about teams is that?

STUCK? TRY THIS

The positions in a row are counted from 0, not from 1. Choosing 2 from 6 is the number in **position 2** of **row 6**.

★ **CHALLENGE ZONE**

Use row 8 to answer: how many ways to choose 3 from 8, and how many to choose 5 from 8?

A row of the triangle adds up to 1,024. Which row is it?

WEEK 29 DOTS & LINES

Trees

Joined up, with no way round. Count the lines and you will find a rule.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

NUMBER WARM-UP · 2 MINUTES
5 dots joined up with no loops. Guess how many lines before you count.

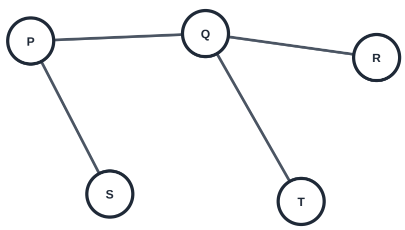
WORDS TO KEEP

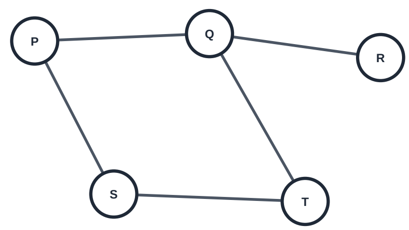
connected you can get from any dot to any other along the lines

cycle a path that leaves a dot and comes back to it without repeating a line

tree connected, and with no cycle anywhere in it

The first drawing is a tree. The second is not, and one extra line is the whole difference.





1 Spot the difference.

a. Find the cycle in the second drawing and write the dots it goes through.

____

b. Dots and lines in the tree: ________ and ________

c. Dots and lines in the other one: ________ and ________

2 Draw your own and count.

Dots in the tree	lines
3	
5	
8	
12	

Write the rule: a tree with **n** dots has ______________________ lines.

3 Why one fewer.

Start with one dot and no lines. Every new dot you add has to be joined on by exactly **one** line, or it would either be cut off or make a cycle. Finish the argument.

____

✎ **ADD TO YOUR PROOF FILE**

Proof File page 27: the rule, and the one-dot-at-a-time argument for why it is always true. It is another invariant, in disguise.

💬 **TALK ABOUT IT**

Why would a second line to a new dot make a cycle?

Is every graph with 5 dots and 4 lines a tree? Draw one that is not.

STUCK? TRY THIS

Add the dots one at a time and watch the count. Each new dot brings exactly one new line with it, which is why the lines always stay one behind.

★ **CHALLENGE ZONE**

A graph has 10 dots and 8 lines. Can it be connected? Use the rule.

A graph has 10 dots and 12 lines and is connected. How many cycles must it have at least?

The Cheapest Network

Join every town for the least money. The answer is always a tree.

IN CLASS THIS WEEK
Multi-step problems (4.OA.3)

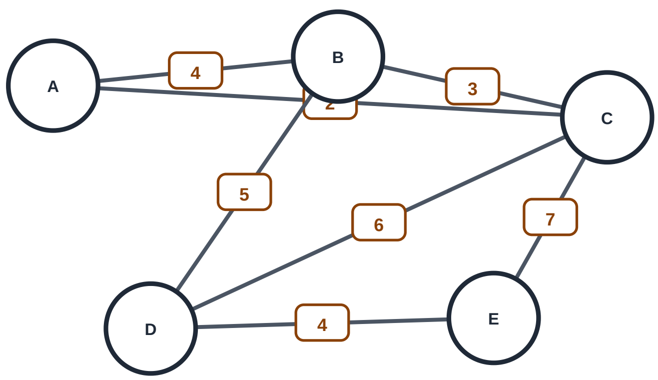
NUMBER WARM-UP · 2 MINUTES
Which is cheaper, 2 + 3 + 4 + 5 or 2 + 3 + 4 + 6?

WORDS TO KEEP

network the set of roads you decide to build

cheapest network joins every town for the smallest total cost

Five towns. The number on each road is what it costs to build. You must be able to get from every town to every other, for as little as possible.



1 Build it cheapest first.

The roads are already listed cheapest first. Take each one **unless** it would make a cycle.

Road	cost	take it?	why not
A to C	2		
B to C	3		
A to B	4		
D to E	4		
B to D	5		
C to D	6		
C to E	7		

2 Check your answer.

a. How many roads did you build? ________ How many towns are there? ________

b. Is that the tree rule from last week? ________

c. Total cost: ________

3 Why a cycle is always a waste.

Suppose your network had a cycle in it. You could delete the dearest road on that cycle and still get everywhere. Say why, and say what that proves about the cheapest network.

____

✎ **ADD TO YOUR PROOF FILE**

Proof File page 28: your network drawn in, its total cost, and the sentence **the cheapest network can never contain a cycle.**

💬 **TALK ABOUT IT**

You never looked at all the possible networks. Why is taking the cheapest road each time good enough?

If two roads cost the same, does it matter which you take first?

STUCK? TRY THIS

Sort the roads by cost before you start. Then take them in order, skipping any road whose two towns are already joined by roads you have taken.

★ **CHALLENGE ZONE**

Change the cost of B to D from 5 to 1 and rebuild. Which roads change?

If every road cost exactly the same, how many different cheapest networks would there be?

Two Ways to Multiply

Two methods, the same answer, and a fair way to compare the work.

IN CLASS THIS WEEK
Multiplying multi-digit numbers (4.NBT.5)

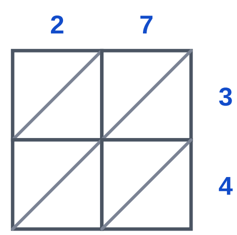
NUMBER WARM-UP · 2 MINUTES
What is 7 × 4? 2 × 3? Those are single-digit multiplications.

WORDS TO KEEP

single-digit multiplication one times-table fact, like 7 × 4. The unit of work for this week

lattice a grid method where every times-table fact gets its own box

The lattice for **27 × 34**. Each box takes one digit from the top and one from the side, and the answer goes in split by the diagonal.



27 × 34

1 Fill it in.

a. How many boxes are there? ________ So how many times-table facts do you need? ________

b. Fill in every box, then add along the diagonals from bottom right.

c. The answer is ________

2 Now the standard way.

a. Work out 27 × 34 the way you were taught in class.

b. Same answer? ________

c. Count the single-digit multiplications you did. ________ Compare with the lattice. ________

3 Predict before you do it.

Problem	boxes in the lattice	single-digit facts
46 × 23		
7 × 385		
124 × 56		

✎ ADD TO YOUR PROOF FILE

Proof File page 29: both methods for 27 × 34 side by side, with the count of times-table facts under each. Same count, different bookkeeping.

💬 TALK ABOUT IT

Neither method does fewer times-table facts than the other. So what is actually different between them?

Which one hides the carrying, and is hiding it a good thing?

STUCK? TRY THIS

Count the boxes before you fill any in. A two-digit number times a two-digit number always needs 2 × 2 boxes, and that count does not depend on the method.

★ CHALLENGE ZONE

Work out 385 × 47 with the lattice, and check it by casting out nines from Week 16.

How many single-digit facts for a three-digit number times a three-digit number? Write the rule for a digits by b digits.

WEEK 32 STEPS & RULES

The Euclid Machine

Take the smaller from the bigger, over and over. It always stops, and where it stops matters.

IN CLASS THIS WEEK
Multi-step problems; factors (4.OA.3, 4.OA.4)

NUMBER WARM-UP · 2 MINUTES
48 − 18 = ? Now take 18 away from your answer.

WORDS TO KEEP

the machine take the smaller number from the bigger, and repeat until the two are equal

terminates a method that is guaranteed to stop, rather than run forever

Run the machine on **48** and **18**. Every row takes the smaller away from the bigger.

bigger	smaller	take it away
48	18	48 − 18 = 30
30	18	30 − 18 = 12
18	12	18 − 12 = 6
12	6	12 − 6 = 6

both are now 6, so the greatest common factor is 6

1 Check it against Week 7.

a. The machine stopped at ________

b. Work out the greatest common factor of 48 and 18 by listing factors, the Week 7 way.

____

c. Same answer? ________ Which method was less work? ________________________

2 Run it three more times.

Start with	subtractions	stops at
35 and 14		
30 and 12		
21 and 13		

One pair stopped at 1. What does that say about those two numbers?

____

3 Why it cannot run forever.

Every row makes one of the two numbers **smaller**, and neither can go below 1. Finish the argument for why the machine always stops.

____

✎ ADD TO YOUR PROOF FILE

Proof File page 30: the machine's steps for 48 and 18, and the reason it must stop. That reason is the same shape as the invariant arguments from Week 1: watch a number that can only go one way.

TALK ABOUT IT

The machine never looks for factors and never divides. How does it find the greatest common factor anyway?

Which pair took the most subtractions? What was unusual about it?

STUCK? TRY THIS

Write the pair on each line and subtract. When the two numbers finally match, that matching number is the answer.

★ CHALLENGE ZONE

Run the machine on 1,071 and 462. It is faster than you expect.

Find the pair below 30 that takes the **most** subtractions. It is not the pair you would guess, and once you see it the reason is obvious.

WEEK 33 FRACTIONS

The Same Point on the Line

Equivalent fractions are not two things that are equal. They are one point with two names.

IN CLASS THIS WEEK
Equivalent fractions (4.NF.1, 4.NF.2)

NUMBER WARM-UP · 2 MINUTES
How many fourths make one half? How many eighths?

WORDS TO KEEP

equivalent different names for the same amount

the same point on one number line, equivalent fractions land on exactly the same mark

One line from 0 to 1, ticked three ways. Read straight **down** a tick to find every name for that point.

1 Read down the ticks.

a. The halfway mark is called ________ halves, ________ fourths and ________ eighths.

b. Three fourths is also ________ eighths.

c. Is there a mark for one third on this line? ________ Why not?

____

2 Multiply top and bottom.

Cutting every piece into 2 doubles the number of pieces **and** the number you take. The point does not move.

Fraction	× 2 top and bottom	× 3 top and bottom
1 over 2	2 over 4	
3 over 4		
2 over 3		
5 over 6		

3 Compare two that do not share a bottom.

Which is bigger, **2 over 3** or **3 over 4**? Rename both with a bottom of 12 first.

______________________ and ______________________, so the bigger one is

____

✎ **ADD TO YOUR PROOF FILE**

Proof File page 31: the number line with the shared ticks marked, and the rule that multiplying top and bottom by the same number never moves the point.

💬 **TALK ABOUT IT**

Why does multiplying the top and bottom by the same number leave the amount alone?

Two fractions land on different points. Can renaming them ever make them equal?

STUCK? TRY THIS

To compare two fractions, rename them both so they have the **same bottom number**. Then the bigger top wins, and it is safe.

★ **CHALLENGE ZONE**

Find three fractions equivalent to 3 over 5, and mark them all on one number line.

Put these in order without a calculator: 2 over 3, 5 over 8, 3 over 4. Pick a bottom number that works for all three.

WEEK 34 FRACTIONS

Parts of a Hundred

A hundred squares, and two ways to say the same shading.

IN CLASS THIS WEEK
Decimal notation for fractions (4.NF.5, 4.NF.6)

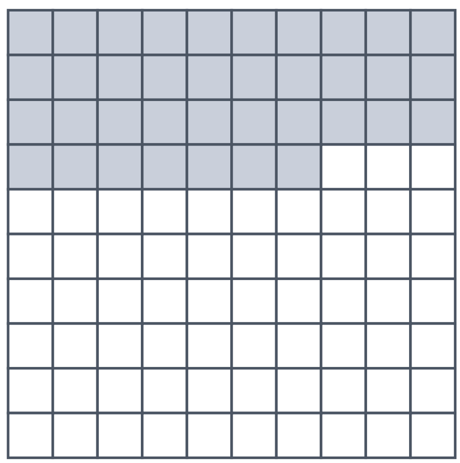
NUMBER WARM-UP · 2 MINUTES
How many hundredths in one tenth? In one half?

WORDS TO KEEP

hundredth one square of a hundred

decimal point everything after it is parts of one, not whole ones

Thirty-seven of the hundred squares are shaded.



37 out of 100

as a fraction

____

as a decimal

____

1 Two names, one picture.

a. As a fraction of a hundred: ____________________

b. As a decimal: ____________________

c. How many **whole rows** are shaded, and how many spare squares? ________ and ________.
So it is ________ tenths and ________ hundredths.

2 Fill in the table.

Shaded squares	fraction of 100	tenths and hundredths	decimal
37	37 over 100	3 and 7	0.37
50			
8			
70			
25			

3 The awkward one.

a. Shade one quarter of the grid. How many squares? ________ As a decimal: ________

b. Now try one **eighth**. How many squares would that be? ________ Can you shade that many whole squares? ________

✎ ADD TO YOUR PROOF FILE

Proof File page 32: the shaded grid with all three names written beside it, and the eighth problem, which is the one worth remembering.

💬 TALK ABOUT IT

Why is 0.7 the same as 0.70 but not the same as 0.07?

Which fractions can be written exactly with a hundred squares, and which cannot?

STUCK? TRY THIS

Count whole rows first. Each whole row is one tenth, and each spare square is one hundredth, which is exactly what the two decimal places mean.

★ CHALLENGE ZONE

Which of these can be shaded exactly on a hundred grid: one half, one third, one fifth, one eighth, one twentieth?

What is special about the bottom numbers that work? Look at their prime factors.

The Weighing Puzzles

How much can one weighing possibly tell you? Count before you weigh.

IN CLASS THIS WEEK
Multi-step problems and reasoning (4.OA.3)

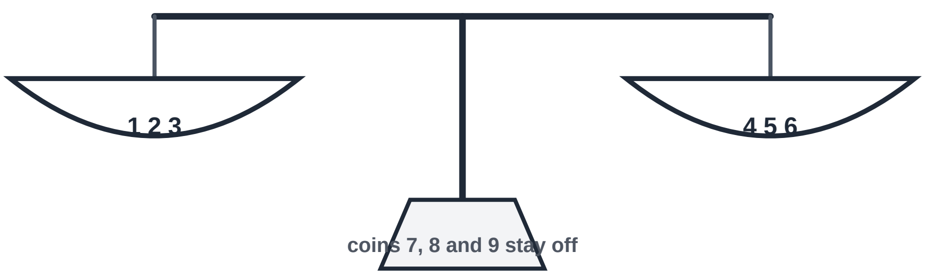
NUMBER WARM-UP · 2 MINUTES
$3 \times 3 = ?$ $3 \times 3 \times 3 = ?$

WORDS TO KEEP

outcome one of the things that can happen. A balance has three: left down, right down, or level

information how many different cases a set of weighings could possibly tell apart

Nine coins. One is heavier than the rest and they all look the same. You have a balance and **two** weighings.



1 Count what a weighing can tell you.

a. How many things can happen when you weigh? ________ Name them.

____

b. So two weighings can tell apart at most $3 \times 3 =$ ________ different cases.

2 Now do it.

a. The picture shows the first weighing. If the left side goes down, which coins is the heavy one among? ____________________

b. If it balances? ____________________

c. Either way you are down to _______ coins. Describe the second weighing.

____

3 How many coins could you handle?

Weighings	cases you can tell apart	most coins
1	3	
2	9	
3		
4		

✎ ADD TO YOUR PROOF FILE

Proof File page 33: the three-outcome count and the table. This is the last new page, and it is the only one where counting tells you what is **possible** rather than what is impossible.

TALK ABOUT IT

Twelve coins and two weighings. Say why it cannot be done, before trying anything.

The counting says three weighings could handle 27 coins. Does that mean a method exists?

STUCK? TRY THIS

Split into **three** equal groups, not two. A balance has three outcomes, so three groups is what uses a weighing fully.

★ CHALLENGE ZONE

Twenty-seven coins, one heavy, three weighings. Describe your first weighing.

Now the hard version: twelve coins, one is the odd weight but you do not know whether it is heavy or light. How many cases is that? Compare with what three weighings can tell apart.

The Proof File, Finished

Thirty-three pages of reasons. This week they become one folder.

IN CLASS THIS WEEK
Review of the year

NUMBER WARM-UP · 2 MINUTES
No warm-up. Get every Proof File page out and put them in order.

Check your file against this list. Every page should have a **claim** and a **reason**, not just an answer.

1 The check list

Weeks	The page should say	✓
1 to 2	invariants; parity rules things out	
3 to 7	factors, primes, trees, common factors and multiples	
8 to 9	coloring, and the cut chessboard	
10 to 11	degrees add to twice the lines; the tracing rule	
14 to 18	divisibility with reasons; casting out nines; the wheel	
19 to 21	fold lines, turn order, the eight moves	
22 to 24	angles add to 180; which shapes tile; impossible tilings	
26 to 28	choosing teams; the triangle; what a row total counts	
29 to 32	trees, cheapest networks, two multiplications, the machine	
33 to 35	one point two names; a hundred squares; counting outcomes	

2 Four last claims.

Prove or disprove. Name the tool.

a. Eleven people can each shake hands with exactly three others. ________

b. 51 is prime. ________

c. A regular octagon tiles a floor on its own. ________

d. A tree with 20 dots has 19 lines. ________

3 Teach one page.

Pick the proof you are proudest of and give it out loud to somebody who has not seen it, without reading. Write down the first thing they did not believe.

____

✎ ADD TO YOUR PROOF FILE

Write a first page for the file: your name, the year, and one sentence saying what a proof is, in your own words. Then number every page and make a contents list.

💬 TALK ABOUT IT

Which was harder this year: showing something is possible, or showing it is not?

Is there a page you can no longer explain? Go back to it before you call the file finished.

STUCK? TRY THIS

A page you cannot say out loud is not finished. That is the only test this week.

★ CHALLENGE ZONE

Look back at Week 1 and Week 29. The cups and the trees are both arguments about a number that can only change one way. Write down what those two weeks have in common.

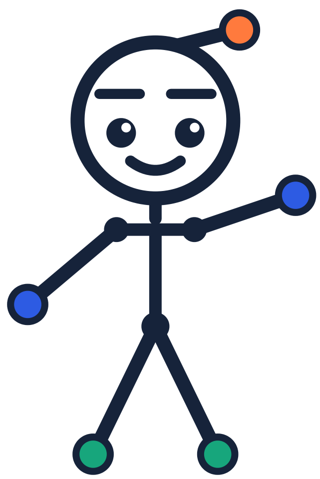
Invent one impossibility of your own, prove it, and put it on the last page of the file.

CERTIFICATE OF COMPLETION

Prover of Impossible Things

awarded to

for a whole year of finding what a move cannot change, coloring a board to settle it, counting degrees, folding and turning shapes, running the machine, and, hardest of all, proving that something *cannot be done at all*.



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