Young Innovators
FOR CHANGE
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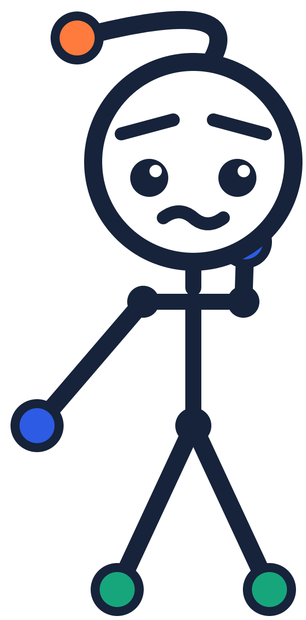
A FULL-YEAR ENRICHMENT PROGRAM · GRADE 6

Find the Rule

36 weekly assignments on turning a pattern into a rule and then proving it, and on the one week where the pattern *lied*



This book belongs to

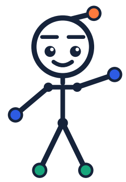
____

Teacher ______________________ School year ______________________

One assignment a week · two pages, front and back · about 30 to 40 minutes
Built to sit alongside the California Grade 6 math curriculum, not to replace it.

How This Book Works

Read this page with a grown-up before Week 1.



Hello for the last time. I am Pip.

Five books ago we counted dots and lines. This year is the one where the patterns become **rules with letters in them**, and then the rules get proved. In Week 4 you finish something you started in Grade 2, and in Week 8 you meet a pattern that fits five cases and is a lie.

One assignment a week, two pages

Page 1 has a note about what you are doing in class this week, a two-minute **Number Warm-Up**, the **Words to keep**, and the main problem.

Page 2 has more to try, then **Add to your Rule Book**, a **Talk About It** question, a hint, and a ★ **Challenge Zone**.

The Rule Book

One folder, built all year, one page per rule. Every page needs the rule **and** a note saying whether you found it by looking, checked it in a few cases, or proved it. Those are three different things and this book is mostly about the difference.

Three rules for this book

1. A letter is not a number you have to find. It is every number at once.
2. Write down how sure you are. **I looked**, **I checked** and **I proved** are all honest answers. Pretending is not.
3. One counterexample beats any number of examples. Week 8 is there to make that stick.

For the grown-up helping

The question worth repeating is **“how do you know that works for every n?”** A child who has checked four cases feels certain, and that feeling is exactly what this year is for correcting. Do not rush the pages where the answer is already obvious to them: the answer was never the point, and the reason takes longer. This is the last book in the line, so it is also worth looking back with them at how far the same few questions have travelled.

The Year at a Glance

Weeks 1–18. The last column is what the class is usually doing that week.

WK	ASSIGNMENT	STRAND	IN CLASS THIS WEEK
TRIMESTER 1 · WEEKS 1–12			
1	**A Letter For Any Number**	Patterns & Rules	Writing and reading expressions (6.EE.2, 6.EE.6)
2	**A Pattern That Grows**	Patterns & Rules	Expressions and equivalent expressions (6.EE.2, 6.EE.4)
3	**Triangular Numbers**	Patterns & Rules	Expressions and equivalent expressions (6.EE.2, 6.EE.3)
4	**The Handshake Formula, Proved**	Ways of Counting	Expressions and equivalent expressions (6.EE.2, 6.EE.4)
5	**Adding One to n**	Patterns & Rules	Expressions; the distributive property (6.EE.2, 6.EE.3)
6	**The Odd Numbers Make a Square**	Patterns & Rules	Expressions and equivalent expressions (6.EE.2)
7	**Looking Back or Jumping Ahead**	Patterns & Rules	Expressions; dependent and independent quantities (6.EE.2, 6.EE.9)
8	**When a Pattern Lies**	Logic & Proof	Expressions and reasoning (6.EE.2, 6.EE.9)
9	**Proof by Picture**	Logic & Proof	Expressions and equivalent expressions (6.EE.3, 6.EE.4)
10	**Proof by Cases**	Logic & Proof	Reasoning about expressions (6.EE.2, 6.EE.5)
11	**Counting the Same Thing Twice**	Ways of Counting	Expressions and equivalent expressions (6.EE.3, 6.EE.4)
12	**Rule Fair**	Patterns & Rules	Writing and evaluating expressions (6.EE.2, 6.EE.6)
TRIMESTER 2 · WEEKS 13–24			
13	**Case Files**	Logic & Proof	Review & reasoning
14	**Ratio as a Rule**	Ratio & Rate	Ratio concepts and reasoning (6.RP.1, 6.RP.3)
15	**One of Them**	Ratio & Rate	Unit rate and rate reasoning (6.RP.2, 6.RP.3)
16	**Out of a Hundred**	Ratio & Rate	Percent as a rate per 100 (6.RP.3c)
17	**Scaling a Network**	Dots & Lines	Ratio and rate reasoning (6.RP.3, 6.EE.2)
18	**Everyone Plays Everyone**	Dots & Lines	Ratio and rate reasoning (6.RP.3, 6.EE.2)

The Year at a Glance

Weeks 19–36. The last column is what the class is usually doing that week.

WK	ASSIGNMENT	STRAND	IN CLASS THIS WEEK
19	**Factors, By Primes**	Number Structure	Greatest common factor and least common multiple (6.NS.4)
20	**All Four Quadrants**	Number Structure	The coordinate plane and negative numbers (6.NS.6, 6.NS.8)
21	**Distance on a Grid**	Shape & Space	The coordinate plane; absolute value (6.NS.8, 6.NS.7)
22	**A Square Circle**	Shape & Space	The coordinate plane; absolute value (6.NS.8, 6.G.3)
23	**Nets and Solids**	Shape & Space	Nets and surface area (6.G.4)
24	**The Formula That Always Holds**	Patterns & Rules	Expressions and reasoning (6.EE.2, 6.G.4)
TRIMESTER 3 · WEEKS 25–36			
25	**Puzzle Fair**	Logic & Proof	Review & reasoning
26	**n Choose k**	Ways of Counting	Expressions and equivalent expressions (6.EE.2, 6.EE.3)
27	**Patterns in the Triangle**	Ways of Counting	Expressions and patterns (6.EE.2, 6.EE.9)
28	**At Least One**	Ways of Counting	Expressions and reasoning (6.EE.2, 6.EE.9)
29	**Three Circles, In Symbols**	Ways of Counting	Expressions and equivalent expressions (6.EE.2, 6.EE.3)
30	**How Likely Is It?**	Ratio & Rate	Ratio and rate reasoning (6.RP.1, 6.RP.3c)
31	**Data Has a Shape**	Data	Statistical questions and distributions (6.SP.1, 6.SP.2)
32	**Center and Spread**	Data	Measures of center and variability (6.SP.3, 6.SP.5)
33	**Same Average, Different Story**	Data	Summarizing data in context (6.SP.5, 6.SP.5c)
34	**Finding the Middle**	Data	Measures of center; problem solving (6.SP.3, 6.EE.2)
35	**Pigeonhole, Grown Up**	Logic & Proof	Reasoning and expressions (6.EE.2, 6.EE.9)
36	**The Rule Book, Finished**	Patterns & Rules	Review of the year

A Letter For Any Number

One letter says what a hundred examples cannot.

IN CLASS THIS WEEK
Writing and reading expressions (6.EE.2, 6.EE.6)

NUMBER WARM-UP · 2 MINUTES
If a bag holds n marbles, how many do two bags hold?

WORDS TO KEEP

variable a letter standing for a number you have not fixed yet

expression a rule written with letters and operations, like 3n + 1

evaluate put a number in for the letter and work it out

Matchstick squares in a row. One square takes 4 sticks. Each new square shares a side, so it only takes 3 more.

Squares	1	2	3	4	5	10	n
Sticks	4	7					

1 Write the rule.

a. Each new square adds ________ sticks.

b. So for **n** squares the rule is ______________________

c. Check it at n = 3 and n = 10 against your table. ________ and ________

2 Read four expressions.

Expression	in words	at $n = 5$
$3n + 1$	three times n, then add 1	
$2n - 3$		
$n(n + 1)$		
$(n + 1) \div 2$		

3 Two rules that look different.

a. Work out $\mathbf{2(n + 3)}$ and $\mathbf{2n + 6}$ at $n = 1$, 4 and 10.

____

b. Same every time? ________ Three cases is not a proof, so say why they must always agree.

____

✎ ADD TO YOUR RULE BOOK

Start the Rule Book. Page 1: the matchstick rule, the table it came from, and the sentence **a letter is not an unknown to be found, it is every number at once.**

TALK ABOUT IT

Why is $3n + 1$ more useful than a table with ten rows in it?

Somebody writes the rule as $4 + 3(n - 1)$. Is that wrong?

STUCK? TRY THIS

Look at the **difference** between one row and the next. If it is always the same number, that number is what multiplies n.

★ CHALLENGE ZONE

Matchstick triangles in a row: 3 sticks for the first, 2 for each after. Write the rule for n triangles.

A rule gives 7, 12, 17, 22 for $n = 1, 2, 3, 4$. Find it, then say what it gives at $n = 100$.

WEEK 2 PATTERNS & RULES

A Pattern That Grows

Draw the shape, count the dots, and let the picture hand you the formula.

IN CLASS THIS WEEK
Expressions and equivalent expressions (6.EE.2, 6.EE.4)

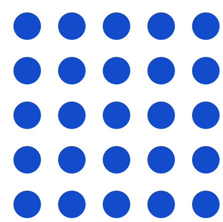
NUMBER WARM-UP · 2 MINUTES
Draw a 4 by 4 square of dots. How many dots without counting them all?

WORDS TO KEEP

figurate number a number you can arrange as a shape: a square, a triangle, a rectangle

the nth term the rule that gives the shape of size n straight away

Square numbers, as squares. The name is not a coincidence.



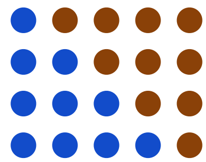
1 From the picture to the rule.

n	1	2	3	4	5	n
dots	1	4				

The rule is ______________________

2 Now an oblong.

Same idea, but **n** rows of **n + 1** dots.



a. At n = 4 that is ________ dots.

b. The rule is ________________________

3 Look at the two shaded halves.

The oblong is split into two staircases of the same size. Each staircase has ________________________ dots, which is next week's whole page.

✎ ADD TO YOUR RULE BOOK

Rule Book page 2: both shapes with their rules underneath. Add the sentence **the arrangement is the argument.**

💬 TALK ABOUT IT

Why is n × n called a square number? Draw the reason rather than saying it.

Could you arrange 12 dots as a square? What does that tell you about 12?

STUCK? TRY THIS

Do not count the dots. Count one row, count the rows, and write what you did with letters instead of numbers.

★ CHALLENGE ZONE

Draw the pentagonal numbers: 1, 5, 12, 22. Find the differences, then the differences of the differences.

A shape has n rows of n + 2 dots. Write the rule and check it at n = 3.

WEEK 3 PATTERNS & RULES

Triangular Numbers

1, 3, 6, 10, 15. You have met them three times already. Now you get the formula.

IN CLASS THIS WEEK

Expressions and equivalent expressions (6.EE.2, 6.EE.3)

NUMBER WARM-UP · 2 MINUTES

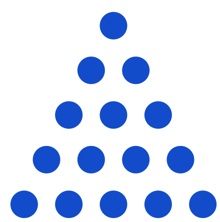
1 + 2 + 3 + 4 + 5 = ? Now add 6.

WORDS TO KEEP

triangular number the dots in a triangle with rows of 1, 2, 3 up to n

closed form a rule that gives the answer in one step, with no adding up

The fifth triangular number, as a triangle. Rows of 1, 2, 3, 4 and 5.



1 Two triangles make an oblong.

a. Take a second copy, turn it upside down, and slot it against the first. What shape do you get? ______________________

b. For n = 5 that oblong is ________ by ________, so it has ________ dots.

c. One triangle is half of it: ________

2 The rule.

a. For any n, the oblong is n by ____________________, so the triangle is ____________________

n	1	2	3	4	5	6	10	100
triangle	1	3						

3 Where you have seen them.

1, 3, 6, 10, 15 appeared in Grade 3 as the third diagonal of the triangle of numbers, and in Grade 2 as something else entirely. Next week says what.

____

✎ ADD TO YOUR RULE BOOK

Rule Book page 3: the two-triangles-make-an-oblong picture and the formula it gives. Do not write the formula without the picture beside it.

💬 TALK ABOUT IT

The formula has a division by 2 in it and always gives a whole number. Why must it?

Which is easier to trust, the picture or the formula? Which is easier to use?

STUCK? TRY THIS

Do not add the rows up. Make the shape into a rectangle you already know how to count, then halve it.

★ CHALLENGE ZONE

What is the 100th triangular number? Use the formula, not addition.

Add two triangular numbers next to each other, like 6 + 10. Do it three times. What do you always get?

The Handshake Formula, Proved

Grade 2 counted them. Grade 3 found the rule. This week you prove it.

IN CLASS THIS WEEK
Expressions and equivalent expressions (6.EE.2, 6.EE.4)

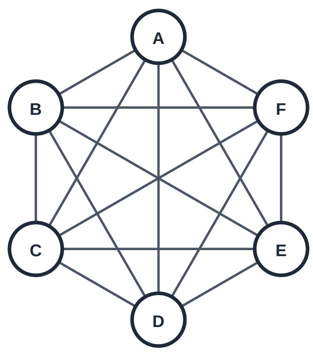
NUMBER WARM-UP · 2 MINUTES
6 people all shake hands. How many handshakes? How did you get it?

WORDS TO KEEP

proof an argument covering every n at once, not a table of cases

double counting counting the same thing two ways and setting the answers equal

Six people, every pair shaking once.



1 Count it two ways.

a. Every person shakes ______________________ hands, and there are n people, so that is ______________________ shakes counted.

b. But each handshake got counted by ________ people.

c. So the real number is ______________________

2 Check it, then look at it.

People	6	10	20	100
handshakes				

a. Compare your formula with last week's triangular formula. What is different?

____

b. So the handshakes for n people are the triangular number for ____________________

3 The same picture, twice.

Draw 5 people in a row and the handshakes as a staircase: the first person shakes 4, the next 3, then 2, then 1. Now you have proved it a second way, by adding.

✎ ADD TO YOUR RULE BOOK

Rule Book page 4: both proofs of the handshake formula, the double count and the staircase. This closes a thread that has been open since Grade 2 week 12.

💬 TALK ABOUT IT

Two proofs of the same rule. Is the second one wasted work?

Which proof would still work if some pairs refused to shake?

STUCK? TRY THIS

Count the ends of the handshakes rather than the handshakes. Every handshake has exactly two ends, which is where the divide by 2 comes from.

★ CHALLENGE ZONE

A tournament where every team plays every other once. 12 teams. How many games?

A room has 66 handshakes. How many people? Solve it with the formula rather than by trying numbers.

WEEK 5 PATTERNS & RULES

Adding One to n

Pair the ends. The whole sum falls out in one line.

IN CLASS THIS WEEK
Expressions; the distributive property (6.EE.2, 6.EE.3)

NUMBER WARM-UP · 2 MINUTES
1 + 10, 2 + 9, 3 + 8. Notice anything?

WORDS TO KEEP

pairing matching the first with the last, the second with the second last, and so on

Gauss the story says he found this at school in about a minute. The story may not be true; the method is

Write 1 to n forwards, and n to 1 underneath. Every column adds to the same thing.

1 up	1	2	3	4	5	6	7	8
n down	8	7	6	5	4	3	2	1
each	9	9	9	9	9	9	9	9

8 columns, each worth 9

1 Read the picture.

a. Every column adds to ________

b. There are ________ columns, so both rows together total ________

c. That counted the sum ________ times, so 1 + 2 + ... + 8 = ________

2 Now with letters.

a. For any n, each column adds to ______________ and there are ________ columns.

b. So 1 + 2 + ... + n = ______________. Compare it with Week 3.

____

3 Use it.

Sum	answer
1 + 2 + ... + 20	
1 + 2 + ... + 100	
1 + 2 + ... + 1000	

✎ ADD TO YOUR RULE BOOK

Rule Book page 5: the pairing picture with n + 1 written under every column. Same formula as Week 3, arrived at from a completely different direction.

💬 TALK ABOUT IT

Weeks 3 and 5 gave the same formula from two different pictures. Does that make it more believable?

What happens to the pairing when n is odd? Try n = 7 and see.

STUCK? TRY THIS

Write both rows out for a small n first and add each column. The number you get is the one that matters, and it is always the same.

★ CHALLENGE ZONE

Add up the even numbers from 2 to 100. Pair them the same way.

Add up 1 + 2 + ... + 100 and then subtract 1 + 2 + ... + 49. What did you just work out, and can you find a quicker route to it?

WEEK 6 PATTERNS & RULES

The Odd Numbers Make a Square

1, then 1 + 3, then 1 + 3 + 5. The answers are square, and the picture says why.

IN CLASS THIS WEEK
Expressions and equivalent expressions (6.EE.2)

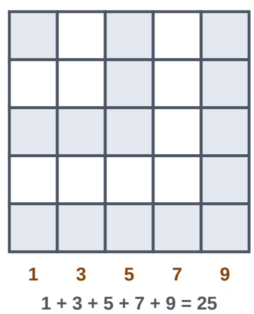
NUMBER WARM-UP · 2 MINUTES
1 + 3 = ?   1 + 3 + 5 = ?   1 + 3 + 5 + 7 = ?

WORDS TO KEEP

gnomon the L-shaped strip you add to a square to make the next square up

proof by picture an arrangement that makes the claim impossible to doubt

A square built from L-shaped strips. Each strip is the next odd number.



1 Count the strips.

a. The strips are 1, 3, 5, 7 and ________

b. There are ________ strips, and together they make a square of side ________

c. So the first n odd numbers add to ______________________

2 Why each strip is the next odd number.

Going from a square of side k to a square of side k + 1 adds a column of ________, a row of ________ and one corner. That is ____________________ squares, which is always odd. Say why it is always odd.

____

3 Use it backwards.

Question	answer
1 + 3 + 5 + ... + 19	
the first 25 odd numbers	
how many odd numbers add to 400	

✎ ADD TO YOUR RULE BOOK

Rule Book page 6: the gnomon square with each strip labeled. A picture that proves something for every n at once is worth more than any number of checks.

💬 TALK ABOUT IT

The picture shows n = 5. How does it prove anything about n = 500?

What do the first n **even** numbers add to? Try the same kind of picture.

STUCK? TRY THIS

Do not add the odd numbers. Count how many of them there are, and square that.

★ CHALLENGE ZONE

1 + 3 + 5 + ... + 99. How many terms, and what is the total?

Find the sum 2 + 4 + 6 + ... + 100 by using this week's rule and one extra step.

Looking Back or Jumping Ahead

Two kinds of rule. One needs the answers before it, and one does not.

IN CLASS THIS WEEK
Expressions; dependent and independent quantities (6.EE.2, 6.EE.9)

NUMBER WARM-UP · 2 MINUTES
A rule says: each answer is the two before it added. Start 1, 2. What comes next?

WORDS TO KEEP

recurrence a rule that builds each answer from earlier ones

closed form a rule that jumps straight to the nth answer without the earlier ones

Two rules, side by side. Both describe the same triangular numbers.

n	1	2	3	4	5	6	20
looking back: add n to the one before	1	3					
jumping ahead: n(n + 1) ÷ 2	1	3					

1 Which is which.

a. To get the 20th answer by looking back, how many steps do you have to do? ________

b. By jumping ahead? ________

2 A rule with no closed form in sight.

The strip coverings from Grade 5: each answer is the two before it added.

n	1	2	3	4	5	6	7	8
coverings	1	2						

Try to find a jumping-ahead rule for these. You will not manage it, and that is the point: some recurrences have one and it is horrible, and some patterns are easier to describe by looking back.

3 Both have their moment.

a. Which kind is better for working out the first ten answers? ____________________

b. Which is better for the thousandth? ____________________

✎ ADD TO YOUR RULE BOOK

Rule Book page 7: both kinds of rule for the triangular numbers, with the number of steps each takes to reach n = 20 written beside it.

TALK ABOUT IT

A looking-back rule needs a starting value. Why does a jumping-ahead rule not?

Grade 5 counted the strip coverings without ever wanting the thousandth. When would you?

STUCK? TRY THIS

Count the work, not the answer. A looking-back rule costs one step per row and a jumping-ahead rule costs one step full stop.

★ CHALLENGE ZONE

Write a looking-back rule for the square numbers, then check it against n squared.

The rule **each answer is double the one before, plus 1**, starting at 1, gives 1, 3, 7, 15, 31. Find a jumping-ahead rule for it. Look at each answer plus one.

WEEK 8 LOGIC & PROOF

When a Pattern Lies

1, 2, 4, 8, 16. Everybody says 32. Everybody is wrong.

IN CLASS THIS WEEK
Expressions and reasoning (6.EE.2, 6.EE.9)

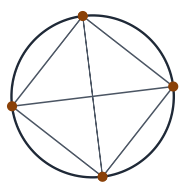
NUMBER WARM-UP · 2 MINUTES
What comes next: 1, 2, 4, 8, 16?

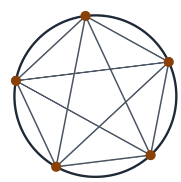
WORDS TO KEEP

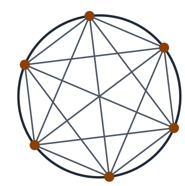
conjecture a guess that fits the cases you have looked at

counterexample one case that kills a conjecture. One is always enough

Points on a circle, every pair joined. Count the **regions** the circle is cut into.


4 points


5 points


6 points

1 Count carefully.

Points	1	2	3	4	5	6
regions	1	2	4			

Count the six-point picture twice. It is not 32, and everybody gets 32 the first time.

2 What went wrong.

a. How many cases fitted the doubling rule before it broke? ________

b. Is four cases a lot? ________

c. Write the lesson in one sentence.

____

3 Two more that look safe.

a. **n** squared + n + 41 gives a prime for n = 0, 1, 2, 3. Try n = 41. ________

b. Every odd number above 1 is a prime plus twice a square. Test 5, 7, 9, 11. It holds a long way and it is false eventually.

✎ ADD TO YOUR RULE BOOK

Rule Book page 8, and give it a red heading: **1, 2, 4, 8, 16, 31.** Every other page in this book is a rule you trusted. This is the page that says why you had to prove them.

💬 TALK ABOUT IT

How many cases would convince you? Is there a number that should?

The Grade 4 book proved things could not be done. What is the connection to this page?

STUCK? TRY THIS

Count the regions by going round the edge, not by eye. Two of the chords in the six-point picture cross at nearly the same place, and that near-miss is exactly where the 32 goes.

★ CHALLENGE ZONE

The real rule for the regions involves choosing 4 points at a time and choosing 2. Look at row 6 of Pascal's triangle and see if you can find 31 in it.

Invent a sequence that looks like it doubles for five terms and then does not.

WEEK 9 LOGIC & PROOF

Proof by Picture

Some arrangements settle a question before a single symbol is written.

IN CLASS THIS WEEK
Expressions and equivalent expressions (6.EE.3, 6.EE.4)

NUMBER WARM-UP · 2 MINUTES
Draw a 3 by 5 rectangle of dots. Now draw a 5 by 3 one. Same count?

WORDS TO KEEP

proof by picture an arrangement that shows a claim for every case at once

what makes it work the picture must be drawable for any n, not just the n you drew

Three claims. For each one, arrange dots or squares until the claim is obvious, then say what you would draw for a general n.

1 Claim one: a rectangle turned is the same rectangle.

Draw 3 by 5 and 5 by 3. What single fact about multiplication does that prove?

2 Claim two: two triangular numbers side by side make a square.

The 3rd triangular number is 6 and the 4th is 10. Add them.

What do you get, and what is the general claim?

3 Claim three: an odd number is always the difference of two squares.

a. 9 = ________ squared − ________ squared

b. Look back at last week's gnomon. A gnomon **is** the difference of two squares. Which two, in general?

____

4 When a picture is not enough.

A picture drawn for n = 4 proves nothing unless you can say how to draw it for any n. Write the test you would apply to somebody else's picture proof.

____

✎ ADD TO YOUR RULE BOOK

Rule Book page 9: all three pictures, each with one line saying how you would draw it for a general n. That line is what turns a drawing into a proof.

TALK ABOUT IT

A picture for n = 4 and a check at n = 4 are not the same thing. What is the difference?

Which of the three claims was hardest to see? Why?

STUCK? TRY THIS

Ask of every picture: could I describe how to draw this for n = 100 without drawing it? If yes, it is a proof. If no, it is an example.

★ CHALLENGE ZONE

Prove with a picture that the sum of the first n even numbers is n(n + 1).

Find a picture proof that 1 + 2 + 4 + 8 + ... + 2 to the n is one less than the next power of 2.

WEEK 10 LOGIC & PROOF

Proof by Cases

Split the question into a few kinds, settle each one, and you have settled them all.

IN CLASS THIS WEEK
Reasoning about expressions (6.EE.2, 6.EE.5)

NUMBER WARM-UP · 2 MINUTES
Every whole number is either odd or even. Is there a third kind?

WORDS TO KEEP

cases a small number of kinds that between them cover everything

exhaustive the cases leave nothing out. If they do, the proof is worthless

Claim: **n** multiplied by **n + 1** is always even, for every whole number n.

Case	then n is	and n + 1 is	so the product is
n is even	even		
n is odd	odd		

1 Finish the argument.

a. In both cases the product is ______________________

b. Are there any other cases? ________

c. So the claim is proved for ______________________ whole numbers.

2 Why last week's formula is safe.

The triangular formula divides $n(n + 1)$ by 2 and always gives a whole number. You have just proved why. Write the connection in one sentence.

3 Two more to prove by cases.

a. The square of an odd number is always odd. Two cases are not needed here, but say which one you use.

b. Any three numbers in a row multiply to a multiple of 3. How many cases do you need, and what are they?

✎ ADD TO YOUR RULE BOOK

Rule Book page 10: the two-case proof written out in full, and the test for a case argument: **do the cases cover everything, and is each one settled?**

💬 TALK ABOUT IT

A proof by cases with a missing case proves nothing. How do you check you have them all?

Could you prove $n(n + 1)$ is even by testing? How many tests would be enough?

STUCK? TRY THIS

Pick the split that makes the claim easy, not the one that comes first to mind. Odd and even is the usual choice, but remainders after 3 or 4 are just as legal.

★ CHALLENGE ZONE

Prove that the difference between two odd numbers is always even.

Prove that $n(n + 1)(n + 2)$ is always a multiple of 6. You will need two different splits.

Counting the Same Thing Twice

Count one thing two ways. The two answers must be equal, so you have a rule for free.

IN CLASS THIS WEEK
Expressions and equivalent expressions (6.EE.3, 6.EE.4)

NUMBER WARM-UP · 2 MINUTES
In a 4 by 6 array of dots, count by rows. Now by columns. Why must they agree?

WORDS TO KEEP

double counting count one collection two different ways and set the answers equal

for free you did not solve anything. You counted twice and the rule appeared

A class of 20, each holding 3 cards. Count the cards two ways.

1 Two counts, one collection.

a. By people: 20 people, 3 cards each = ________

b. If instead there are 4 kinds of card and each kind is held 15 times, that is ________

c. Both count the same cards, so ______________________

2 Where you have already used it.

The collection	counted one way	counted the other way
ends of handshakes	n people, $n - 1$ each	
ends of lines in a graph	add up the degrees	
dots in an oblong	n rows of $n + 1$	

All three of those were double counts. The middle one is the Grade 4 handshake lemma.

3 Do a new one.

In Pascal's triangle, row 6 adds to 64. Count the row two ways: as choosings of 0, 1, 2 up to 6 from six children, and as each child deciding to come or stay home. Write both counts and set them equal.

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 11: the double counting method and the three places this line of books has already used it without naming it.

TALK ABOUT IT

You proved a rule without solving anything. Does that feel like cheating?

What has to be true about the two counts for the trick to be legal?

STUCK? TRY THIS

Name the collection first, out loud, before counting anything. Most double counting goes wrong because the two counts are counting slightly different things.

★ **CHALLENGE ZONE**

A tournament has 45 games and everyone plays everyone once. Use double counting to find the number of teams.

Every one of 30 students joins exactly 2 of 5 clubs, and every club has the same number of members. How many in each club?

Rule Fair

Six patterns. Find each rule, and say how sure you are.

IN CLASS THIS WEEK
Writing and evaluating expressions (6.EE.2, 6.EE.6)

NUMBER WARM-UP · 2 MINUTES
No warm-up. Read all six first.

WORDS TO KEEP
confidence found by looking, checked at a few n, or proved. Three different things

For each pattern write the rule for the nth term, then mark it **L** if you only looked, **C** if you checked cases, or **P** if you can prove it.

Pattern	1st	2nd	3rd	4th	5th	rule for n	L C P
A	3	7	11	15	19		
B	1	4	9	16	25		
C	2	6	12	20	30		
D	1	3	7	15	31		
E	1	5	14	30	55		
F	1	2	4	8	16		

1 Two of them deserve suspicion.

a. Which pattern is the one from Week 8? ________ What would you need before you trusted it?

____

b. Pattern E is the running total of the square numbers. Its rule exists and is not obvious. Did you mark it L?________

2 Prove one of them.

Pick whichever pattern you marked P and write the proof out properly, with a picture if you have one.

____

____

✎ ADD TO YOUR RULE BOOK

Rule Book page 12: the six rules with their L, C or P beside them. An honest C is worth more than a dishonest P.

💬 TALK ABOUT IT

Which of the three marks did you use most? Is that a problem?

Pattern C is twice pattern... something. Which, and does that make it easier to prove?

STUCK? TRY THIS

Take the differences between one term and the next. If those are constant, the rule multiplies n. If the differences of the differences are constant, n squared is involved.

★ CHALLENGE ZONE

Find the rule for pattern E. Look up the sum of the first n square numbers if you must, then check it at n = 5.

Invent a pattern whose first five terms are 1, 2, 4, 8, 16 and whose sixth term is 26. Any rule you can write down counts.

Case Files

Everything from the first twelve weeks, in one folder.

IN CLASS THIS WEEK
Review and reasoning

NUMBER WARM-UP · 2 MINUTES
No warm-up this week.

Five cases. Each needs a rule, and a note on whether you looked, checked or proved.

1 The matchsticks

Hexagons in a row, sharing sides. 6 sticks for the first. How many for n?

2 The tournament

15 teams, everyone plays everyone once. How many games, and which formula did you use?

3 The odd sum

1 + 3 + 5 + ... + 41. How many terms, and what is the total?

4 The suspicious pattern

A sequence starts 2, 4, 8, 16. Somebody says the tenth term is 1024. What would you say back?

____

5 The two cases

Prove that the sum of any three numbers in a row is a multiple of 3.

____

Which tool?

Beside each case write its method: differences ________, the handshake formula ________, the gnomon square ________, counterexample ________, proof by cases ________.

✎ **ADD TO YOUR RULE BOOK**

Add a Rule Book page called **My methods so far**: differences, pictures, pairing, double counting, cases, and the counterexample. One line each.

💬 **TALK ABOUT IT**

Which case could you settle for every n, and which only for the n asked?

Case 4 is the whole first trimester in one question. What is it really testing?

STUCK? TRY THIS

Ask of every case whether you are being asked for **one** answer or for a rule that covers all of them. The two need completely different work.

★ **CHALLENGE ZONE**

Write a sixth case that looks like it needs a formula and actually needs a counterexample. Include an answer key on a separate sheet.

Ratio as a Rule

Three cups of flour to two of water. That is a rule with a letter hiding in it.

IN CLASS THIS WEEK
Ratio concepts and reasoning (6.RP.1, 6.RP.3)

NUMBER WARM-UP · 2 MINUTES
A recipe uses 3 cups of flour to 2 of water. How much water for 6 cups of flour?

WORDS TO KEEP

ratio a comparison of two amounts, written 3 to 2

equivalent ratios 3 to 2 and 6 to 4 and 9 to 6 all say the same thing

Three flour to two water. Scale it up and the ratio does not change.

Flour	3	6	9	12		n
Water	2				20	

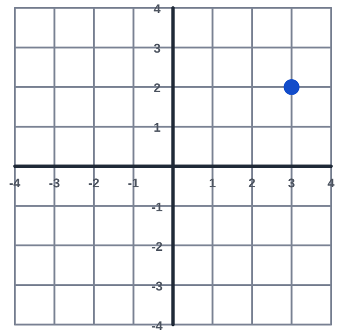
1 Read the table.

a. Water is always ______________________ of the flour.

b. So for n cups of flour, water = ______________________

c. For 20 cups of water you need ________ cups of flour.

2 Plot the pairs.



The points lie on a ______________________ through ______________________

3 Which of these are the same ratio?

Ratio	same as 3 to 2?	how you know
6 to 4		
9 to 5		
15 to 10		
1.5 to 1		

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 13: the ratio table, the rule with n in it, and the plotted line. A ratio is a rule, and the line is what makes that visible.

💬 **TALK ABOUT IT**

Why does the line go through the origin? What would it mean if it did not?

Is 3 to 2 the same as 2 to 3? Say why the order matters.

STUCK? TRY THIS

Find what you multiply the first number by to get the second. That multiplier is the rule, and it does not change however far you scale.

★ **CHALLENGE ZONE**

A paint mix is 5 parts blue to 3 parts white. How much white for 35 blue?

Two mixes: 4 to 6 and 6 to 9. Are they the same? Prove it two different ways.

One of Them

Reduce everything to one, and any two things become comparable.

IN CLASS THIS WEEK
Unit rate and rate reasoning (6.RP.2, 6.RP.3)

NUMBER WARM-UP · 2 MINUTES
12 pencils cost $3. What does one pencil cost?

WORDS TO KEEP

unit rate how much for exactly one

comparable two offers in different sizes cannot be compared until both are per one

Four shops, four sizes, four prices. Which is the best value?

Shop	pencils	price	price for one
A	12	$3.00	
B	20	$4.60	
C	8	$2.08	
D	30	$7.50	

1 Work each one out.

a. Cheapest per pencil: ______________________

b. Was that the shop with the lowest price on the shelf? ________ Or the biggest pack? ________

2 The other unit.

a. Instead of price for one pencil, work out **pencils for one dollar** at each shop.

____

b. Does that pick the same winner? ________ Which unit is easier to think in?

____

3 Rates that are not prices.

Situation	the unit rate
180 miles in 3 hours	
$45 for 5 hours of work	
240 words in 4 minutes	

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 14: the four shops with both unit rates worked out, and the note that both orderings agree. They always do, and it is worth saying why.

TALK ABOUT IT

Price per pencil and pencils per dollar rank the shops the same way. Why must they?

A shop that is cheapest per pencil might still be a bad buy. When?

STUCK? TRY THIS

Divide. Always divide the thing you want one of into the other. Which way round decides which unit rate you get, and both are useful.

★ **CHALLENGE ZONE**

A car does 300 miles on 12 gallons and another does 245 on 10. Which is better, and by how much per gallon?

Shop E sells 25 pencils for $6.00. Where does it rank against the four above?

WEEK 16 RATIO & RATE

Out of a Hundred

Percent is a ratio wearing a uniform. Everything is out of a hundred.

IN CLASS THIS WEEK
Percent as a rate per 100 (6.RP.3c)

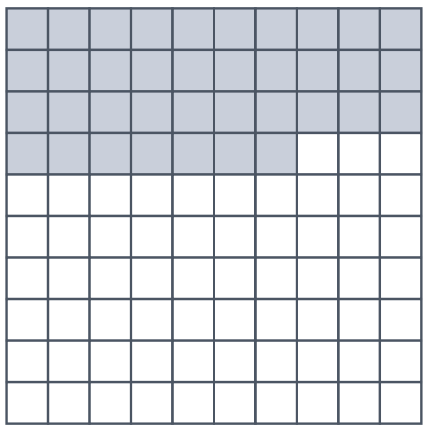
NUMBER WARM-UP · 2 MINUTES
What is 25 out of 100 as a percent? What is 1 out of 4?

WORDS TO KEEP

percent a ratio where the second number is always 100

why 100 because two ratios with the same second number can be compared at a glance

A hundred squares. Shade any number and you have written a percent.



37 out of 100

as a fraction ______

as a decimal ______

1 Three names again.

a. As a fraction of a hundred: ______________________

b. As a decimal: ______________________

c. As a percent: ______________________

2 Compare things that are not out of a hundred.

Result	as a fraction	out of 100	percent
18 out of 25			
21 out of 30			
34 out of 40			
7 out of 8			

Best result: ____________________

3 Percent of a number.

a. 25 percent of 60 = _______ b. 10 percent of 250 = _______

c. A $40 coat is 15 percent off. The discount is _______ and you pay _______

✎ ADD TO YOUR RULE BOOK

Rule Book page 15: the grid with its three names, and the sentence **percent exists so that different-sized results can be compared without thinking.**

💬 TALK ABOUT IT

Why is comparing 18 out of 25 with 21 out of 30 hard until you convert both?

Can a percent be more than 100? What would that mean?

STUCK? TRY THIS

Scale the second number to 100 and do the same to the first. If the second number does not divide 100, divide and multiply by 100 instead.

★ CHALLENGE ZONE

Which is the better score: 43 out of 50 or 60 out of 70?

A price rises by 20 percent and then falls by 20 percent. Is it back where it started? Work it out on $100.

WEEK 17 DOTS & LINES

Scaling a Network

Every road doubles in price. What happens to the cheapest route?

IN CLASS THIS WEEK
Ratio and rate reasoning (6.RP.3, 6.EE.2)

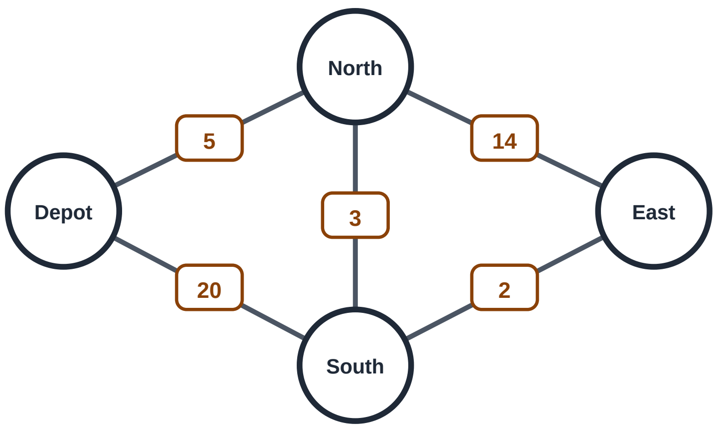
NUMBER WARM-UP · 2 MINUTES
If everything on a menu doubles in price, does the cheapest meal change?

WORDS TO KEEP

scaling multiplying every number in a problem by the same amount

what survives scaling which answers change and which ones do not

A delivery network. Each road shows its cost in dollars.



1 Find the cheapest route.

a. Cheapest from Depot to East: ______________________ costing ________

b. How many roads does it use? ________

2 Now scale everything.

Every road	cheapest cost	the route itself
as printed		
doubled		
multiplied by 10		
plus 10 each		

Three of those keep the same route. One does not. Which, and why?

3 Why adding is different.

Multiplying every road by the same number keeps the ratios between routes. Adding the same amount to every road does not, because a route with more roads picks up more additions. Say that in your own words.

✎ ADD TO YOUR RULE BOOK

Rule Book page 16: the network, the cheapest route, and the four scalings. The plus-5 row is the page.

💬 TALK ABOUT IT

Which quantity is the ratio here, and which is the rate?

If a road were free, would multiplying still keep the route the same?

STUCK? TRY THIS

Work out the cheapest route once, then ask what each scaling does to **every** route before you recompute anything.

★ CHALLENGE ZONE

Multiply every road by one half. What is the cheapest cost now, and is the route the same?

The plus-10 row changed the route. Work out the smallest amount you could add to every road that still changes it.

WEEK 18 DOTS & LINES

Everyone Plays Everyone

A round robin, counted with the handshake formula and read as ratios.

IN CLASS THIS WEEK
Ratio and rate reasoning (6.RP.3, 6.EE.2)

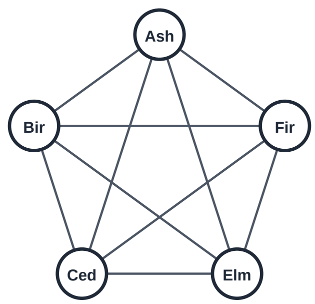
NUMBER WARM-UP · 2 MINUTES
5 teams, everyone plays everyone once. How many games?

WORDS TO KEEP

round robin every team plays every other team exactly once

win ratio wins compared with games played, which is what lets you compare teams

Five teams. Everyone plays everyone once, and there are no draws.



1 Count the games.

a. Each team plays ________ games.

b. Five teams times that is ________, but every game was counted ________ times.

c. So there are ________ games. That is Week 4 again.

2 The table.

Team	wins	games	win ratio	percent
Ash	4			
Birch	3			
Cedar	2			
Elm	1			
Fir	0			

Add the wins column. ________ Should it equal the number of games? ________ Why?

3 A bigger league.

a. 12 teams, round robin. Games: ________

b. A team wins 8 of them. Win percent: ________

✎ ADD TO YOUR RULE BOOK

Rule Book page 17: the games formula, the win ratios, and the check that the wins add up to the games. That check is a double count.

TALK ABOUT IT

Why must the total wins equal the total games?

Two teams in leagues of different sizes both won 6. Can you say which did better?

STUCK? TRY THIS

Count the games with the handshake formula rather than by listing. Then every team's games played is the same number, which makes the ratios easy to compare.

★ CHALLENGE ZONE

A round robin has 66 games. How many teams?

In a 12 team round robin, is it possible for every team to win the same number of games? Work out what that number would have to be.

WEEK 19 NUMBER STRUCTURE

Factors, By Primes

Grade 4 found common factors by listing. Primes make it a rule instead.

IN CLASS THIS WEEK
Greatest common factor and least common multiple (6.NS.4)

NUMBER WARM-UP · 2 MINUTES
Write 24 and 36 as products of primes.

WORDS TO KEEP

prime factorization the primes a number is built from, with how many of each

greatest common factor take the primes both share, the smaller count of each

least common multiple take every prime either uses, the larger count of each

Two numbers, broken into primes. Everything about their factors is in these two rows.

Number	as primes	2s	3s	5s
24	2 × 2 × 2 × 3	3	1	0
36	2 × 2 × 3 × 3			

1 Read both answers off the table.

a. Greatest common factor: take the **smaller** count of each prime. 2s: ________ 3s: ________ 5s: ________ so the answer is ________

b. Least common multiple: take the **larger** count of each. ________

2 Three more pairs.

Pair	greatest common factor	least common multiple	the two multiplied
12 and 18			
15 and 28			
20 and 50			

3 A rule falls out.

For each row, multiply your two answers together and compare with the two starting numbers multiplied. What do you find, and why does the prime table explain it?

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 18: the prime table method for both, and the rule that the greatest common factor times the least common multiple equals the two numbers multiplied.

TALK ABOUT IT

Grade 4 listed every factor. Why is the prime method better for big numbers?

What does the table look like when the greatest common factor is 1?

STUCK? TRY THIS

Write the primes in columns, one column per prime, and fill in zero where a number does not use that prime. Then the two answers are just smaller-of and larger-of.

★ **CHALLENGE ZONE**

Find the greatest common factor and least common multiple of 84 and 126.

Two numbers multiply to 3,600 and their greatest common factor is 12. What is their least common multiple?

All Four Quadrants

The grid goes left and down as well, and nothing else changes.

IN CLASS THIS WEEK
The coordinate plane and negative numbers (6.NS.6, 6.NS.8)

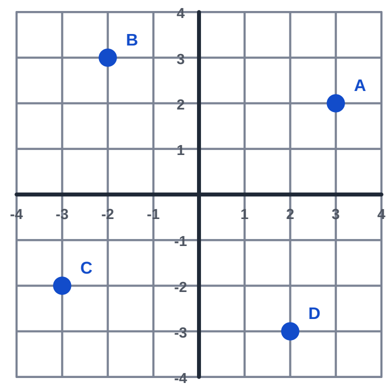
NUMBER WARM-UP · 2 MINUTES
If right is positive, what is left? If up is positive, what is down?

WORDS TO KEEP

quadrant one of the four regions the two axes cut the plane into

reflection flipping a point across an axis changes the sign of one coordinate

Four points, one in each quadrant.



1 Read them off.

Point	across	up	quadrant	signs
A	3	2		+ +
B				
C				
D				

2 Flip them.

a. Reflect A across the up-down axis. New point: ____________________ Which coordinate changed sign? ____________________

b. Reflect A across the left-right axis. ____________________

3 Distance along the grid lines.

a. How far apart are A(3, 2) and B(-2, 3) going across? _______ And up and down? _______

b. Careful with the signs: the distance is the difference, and a difference is never negative.

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 19: the four quadrants with their sign patterns, and the reflection rule. Next week uses the across-and-up distance for something surprising.

💬 **TALK ABOUT IT**

Why does reflecting across the up-down axis change the **across** coordinate?

Which quadrant has both coordinates negative? Why is it numbered third?

STUCK? TRY THIS

Read the across number first, always. Getting (3, 2) and (2, 3) confused is the one mistake this page is really about.

★ **CHALLENGE ZONE**

Plot (-4, 1), reflect it across both axes in turn, and write the three points you get.

Two points have the same across coordinate. What do you know about the line joining them?

WEEK 21 SHAPE & SPACE

Distance on a Grid

A delivery van cannot go through buildings. Its distance is a different distance.

IN CLASS THIS WEEK
The coordinate plane; absolute value (6.NS.8, 6.NS.7)

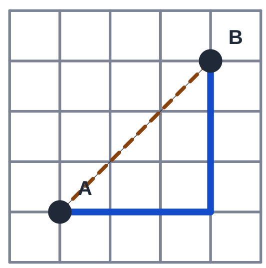
NUMBER WARM-UP · 2 MINUTES
From (1, 1) to (4, 4). How far is that if you must stay on the streets?

WORDS TO KEEP

taxicab distance across plus up, because you must follow the streets

straight line distance what a bird gets. Always less than or equal to the taxicab distance

A(1, 1) to B(4, 4). The solid route follows streets. The dashed one is what a bird would fly.



along the streets: 6 blocks

1 Count the blocks.

a. Across: ________ Up: ________ Total: ________

b. Try a different street route from A to B. How many blocks? ________

c. Every street route from A to B is the same length. Say why.

____

2 The rule.

For points (a, b) and (c, d), the taxicab distance is ______________________ plus ______________________, where each difference is taken as a positive number.

3 Four pairs.

From	to	taxicab distance
(0, 0)	(3, 4)	
(2, 5)	(2, 1)	
(-1, 2)	(3, -1)	
(4, 4)	(1, 1)	

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 20: the two routes drawn on one grid, the taxicab rule, and the reason every street route between two points has the same length.

💬 **TALK ABOUT IT**

Grade 5 counted routes across a grid and Grade 6 measures them. What is the connection?

When are the taxicab distance and the straight line distance equal?

STUCK? TRY THIS

Take the two across numbers and subtract, ignoring which is bigger. Do the same up and down. Add the two.

★ **CHALLENGE ZONE**

Which points are exactly 4 blocks from (0, 0)? List them all. There are more than you think.

Find two points where the taxicab distance is 10 and the straight line distance is exactly 10 as well.

WEEK 22 SHAPE & SPACE

A Square Circle

A circle is every point the same distance from the middle. Change the distance, change the circle.

IN CLASS THIS WEEK
The coordinate plane; absolute value (6.NS.8, 6.G.3)

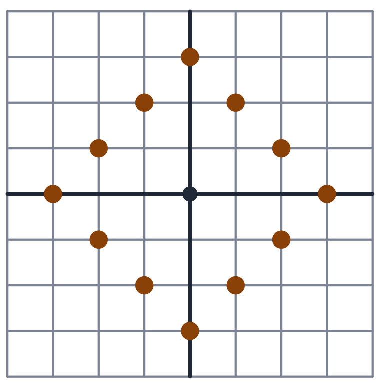
NUMBER WARM-UP · 2 MINUTES
Name three points exactly 3 blocks from (0, 0), traveling on streets only.

WORDS TO KEEP

circle every point exactly r away from a chosen middle. The definition never mentions round

it depends on the distance change what distance means and the same definition draws a different shape

Every point exactly **3 blocks** from the middle, going along streets.



every point exactly 3 blocks from the middle

1 What shape is it?

a. How many points are exactly 3 blocks away? ________

b. What shape do they make? ________________________

c. Is it a circle? Read the definition again before you answer.

____

2 Count them for other radii.

Radius	1	2	3	4	10	r
points	4					

Write the rule for radius r. ____________________

3 Why the definition matters.

The taxicab circle and the round one come from exactly the same words. Only **distance** changed. Write one sentence about what that tells you about definitions.

____

✎ ADD TO YOUR RULE BOOK

Rule Book page 21: the diamond, the counting rule for radius r, and the sentence **the definition did not change. Only what distance means changed.**

💬 TALK ABOUT IT

Is the taxicab circle really a square? Look at how it sits on the grid.

In taxicab geometry, how many points are the same distance from two given points?

STUCK? TRY THIS

Work outward from the middle. For radius r, count the points on each of the four arms and then the corners, and check for double counting.

★ CHALLENGE ZONE

Draw the taxicab circle of radius 4 and count its points. Does your rule hold?

Two shops are at (0, 0) and (4, 0). Which points are equally far from both, in taxicab distance? The answer is not a straight line.

WEEK 23 SHAPE & SPACE

Nets and Solids

Flatten a solid, count its parts, and put it back together.

IN CLASS THIS WEEK
Nets and surface area (6.G.4)

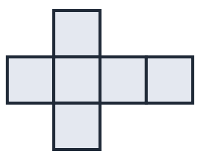
NUMBER WARM-UP · 2 MINUTES
How many faces does a cube have? How many edges?

WORDS TO KEEP

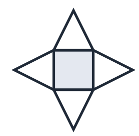
net a flat shape that folds up into a solid

face, edge, corner the flat parts, the lines where two faces meet, and the points where edges meet

Two nets. Fold each one in your head, then count.



a net for a cube: 6 squares



a net for a square pyramid: 1 square, 4 triangles

1 Count all three.

Solid	faces	edges	corners
cube	6		
square pyramid			
triangular prism			

2 Count edges without counting.

a. A cube has 6 square faces, each with 4 sides, so that is ________ sides in all.

b. But every edge is where ________ faces meet, so it got counted that many times.

c. Edges = ________. That is double counting again.

3 Not every arrangement folds.

Six squares in a 2 by 3 block does not fold into a cube. Cut six squares out and try it. Then say what goes wrong.

____

✎ ADD TO YOUR RULE BOOK

Rule Book page 22: the three solids with their counts, and the face-sides-divided-by-2 method for the edges. Next week does something with those three columns.

TALK ABOUT IT

Why does the same solid have several different nets?

A solid has 8 faces. Can you say how many edges without knowing anything else?

STUCK? TRY THIS

Count the faces from the net, because they are just the pieces. Count the edges by counting the sides of every face and halving, because every edge is shared by two faces.

★ CHALLENGE ZONE

How many different nets fold into a cube? There are more than five.

Count the faces, edges and corners of a hexagonal prism using the halving method.

WEEK 24 PATTERNS & RULES

The Formula That Always Holds

Faces, edges and corners. One little sum comes out the same for every solid you try.

IN CLASS THIS WEEK
Expressions and reasoning (6.EE.2, 6.G.4)

NUMBER WARM-UP · 2 MINUTES
A cube has 6 faces, 12 edges and 8 corners. What is 6 + 8 − 12?

WORDS TO KEEP

Euler's formula corners − edges + faces = 2, for every solid without a hole through it

invariant the Grade 4 word. A number that comes out the same however much the thing changes

Fill the table from last week, and work out the last column for each.

Solid	corners C	edges E	faces F	C − E + F
cube	8	12	6	
square pyramid				
triangular prism				
tetrahedron	4	6	4	
octahedron	6	12	8	

1 What do you notice?

a. Every row gives ________

b. Five cases. Is that a proof? ________ Which week said so?______________________

c. It is in fact true for every solid without a hole through it, and the proof is beyond this book. Believing it is not the same as having proved it, and this page says which one you are doing.

2 Use it as a rule.

A solid has	so the missing number is
12 faces and 30 edges	
8 corners and 12 edges	
20 faces and 12 corners	

3 Where it fails.

A solid ring, like a picture frame with a hole through it, gives 0 rather than 2. Does that break the formula, or was the formula always about a particular kind of solid?

✎ ADD TO YOUR RULE BOOK

Rule Book page 23: the five rows, the constant 2, and an honest note that you have checked this and not proved it. Week 8 is the reason that note has to be there.

💬 TALK ABOUT IT

Five cases fitted 1, 2, 4, 8, 16 too. Why is this one different, if it is?

The formula is an invariant. What is being changed and what is staying the same?

STUCK? TRY THIS

Count corners and faces from the net and get the edges by halving the face sides. Then do the little sum, being careful with the minus.

★ CHALLENGE ZONE

Work out C − E + F for a pentagonal prism and a hexagonal pyramid.

A football is made of 12 pentagons and 20 hexagons. Use the halving method for the edges, then use the formula to find the corners.

WEEK 25 LOGIC & PROOF

Puzzle Fair

A little of everything from Weeks 14 to 24.

IN CLASS THIS WEEK
Review and reasoning

NUMBER WARM-UP · 2 MINUTES
No warm-up. Start wherever you like.

Six puzzles. Answer plus method, every time.

1 The mix

Paint is 7 parts blue to 4 parts white. How much white goes with 63 blue?

2 The better buy

15 apples for \$4.50 or 22 for \$6.16. Which is cheaper per apple, and by how much?

3 The primes

Greatest common factor and least common multiple of 45 and 60, using the prime table.

4 The van

Taxicab distance from (-3, 4) to (2, -1). And the straight line distance, roughly.

5 The solid

A solid has 9 faces and 16 edges. How many corners?

6 The league

A round robin has 28 games. How many teams, and how many games does each play?

✎ **ADD TO YOUR RULE BOOK**

Add a Rule Book page called **My methods, weeks 14 to 24**: ratio tables, unit rates, percent, the prime table, taxicab distance, and the faces-edges-corners formula.

TALK ABOUT IT

Puzzle 6 uses a formula from Week 4. Which one, and did you have to solve it backwards?

Which puzzle used a rule you have proved, and which used one you have only checked?

STUCK? TRY THIS

Two of these are the handshake formula in disguise. Working out which two is most of the job.

★ **CHALLENGE ZONE**

Write a seventh puzzle that needs a ratio and the handshake formula together. Include an answer key on a separate sheet.

n Choose k

The counting you have done since Grade 3, finally written as a formula.

IN CLASS THIS WEEK
Expressions and equivalent expressions (6.EE.2, 6.EE.3)

NUMBER WARM-UP · 2 MINUTES
How many ways to choose 3 children from 5? You worked this out in Grade 4.

WORDS TO KEEP

n choose k the number of ways to pick k things from n when order does not matter

the formula multiply n by n − 1 by n − 2, k times over, then divide by k × (k − 1) × ... × 1

Choosing 3 from 5, in two steps, as Grade 4 did it.

1 Build the formula.

a. Line-ups of 3 from 5: 5 × 4 × 3 = ________

b. Each team of 3 appears in 3 × 2 × 1 = ________ line-ups.

c. So 5 choose 3 = ________

d. For any n and k, n choose k =

____

2 Use it.

n choose k	top	bottom	answer
6 choose 2	6 × 5	2 × 1	
7 choose 3			
10 choose 2			
8 choose 4			

3 Two you already knew.

a. n choose 2 = ________________________. Where have you seen that this year?

____

b. n choose 1 = ________ and n choose 0 = ________. Say why the second one is 1 and not 0.

____

✎ ADD TO YOUR RULE BOOK

Rule Book page 24: the formula with the over-count-then-divide reason underneath. The n choose 2 line closes the handshake thread for the last time.

💬 TALK ABOUT IT

Why divide rather than subtract? Say it in terms of line-ups.

n choose 0 is 1. What is the one way to choose nothing?

STUCK? TRY THIS

The top and the bottom always have the **same number** of things multiplied together. If they do not, you have made a mistake.

★ CHALLENGE ZONE

Work out 12 choose 3 and 12 choose 9. Then say why they had to be equal.

A pizza shop has 9 toppings and you pick 4. How many pizzas?

WEEK 27 WAYS OF COUNTING

Patterns in the Triangle

Every row, every diagonal and every sum in Pascal's triangle means something.

IN CLASS THIS WEEK
Expressions and patterns (6.EE.2, 6.EE.9)

NUMBER WARM-UP · 2 MINUTES
Row 5 of the triangle is 1, 5, 10, 10, 5, 1. What is 5 choose 2?

WORDS TO KEEP

row n the choosings from n things, from choose 0 to choose n

symmetry n choose k always equals n choose (n − k)

The triangle again, with what each entry means written under it.

row 0	1							
row 1	1	1						
row 2	1	2	1					
row 3	1	3	3	1				
row 4	1	4	6	4	1			
row 5	1	5	10	10	5	1		
row 6	1	6	15	20	15	6	1	
row 7	1	7	21	35	35	21	7	1

1 Three patterns, three reasons.

a. Each row reads the same backwards. In choosing language that says n choose k equals ______________________. Why, in terms of who you leave behind?

____

b. Row n adds to ______________________. Why, in terms of each person deciding to come or stay?

____

2 The building rule, in choosing language.

Each entry is the two above it added. In choosing language: to choose k from n, either you take the newest person and choose ______________________ from the rest, or you leave them and choose ______________________ from the rest. That is the whole proof of the building rule.

3 The third diagonal.

a. Read down the third entry of each row: 1, 3, 6, 10, 15, 21. What are those?

____

b. And what does the third entry of row n mean in choosing language?
______________________ So two things you met years apart are the same thing.

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 25: eight rows of the triangle with the three patterns and the reason for each. The building rule proof is the best paragraph in the book.

TALK ABOUT IT

The building rule was a pattern in Grade 4 and is a proof now. What changed?

Where in the triangle are the handshake numbers, and where are the row totals?

STUCK? TRY THIS

Whenever you see a pattern in the triangle, ask what it says about **choosing**. Every one of them has a reason in those terms, and the reason is always short.

★ **CHALLENGE ZONE**

Add up the first four entries of the third diagonal: 1 + 3 + 6 + 10. Now find that total somewhere in the triangle. Try a few more.

Row 10 adds to 1024. Which entry is the biggest, and roughly what fraction of the row is it?

At Least One

Three words that always mean the same instruction: count them all, count the ones with none, subtract.

IN CLASS THIS WEEK
Expressions and reasoning (6.EE.2, 6.EE.9)

NUMBER WARM-UP · 2 MINUTES
How many four-digit numbers are there altogether?

WORDS TO KEEP

restriction a condition the things you are counting must satisfy

the complement everything that fails the restriction, which is usually far easier to count

A four-digit code, each digit 0 to 9, and the first digit may be 0. How many contain **at least one** 7?

1 Both counts.

a. Codes altogether: ________ × ________ × ________ × ________ = ________

b. Codes with **no** 7: each digit has ________ choices, so ________

c. At least one 7: ________

2 Choosing with a restriction.

A team of 4 is chosen from 10 people. Two of them are Ana and Ben.

Question	count	how
all teams of 4		10 choose 4
teams with neither Ana nor Ben		8 choose 4
teams with at least one of them		
teams with both of them		8 choose 2

3 Why the complement is usually easier.

Counting **at least one** directly means counting one, and two, and three, and adding. Counting **none** is a single multiplication. Say that in your own words.

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 26: the phrase **at least one** and the three-step instruction it should trigger, with both examples worked.

💬 **TALK ABOUT IT**

When would counting at least one directly actually be easier?

Teams with at least one of Ana and Ben, plus teams with neither. What must those add to?

STUCK? TRY THIS

Write down what **none** means before you count anything. That sentence is usually the whole problem, and the arithmetic after it is one line.

★ **CHALLENGE ZONE**

How many four-digit codes contain at least one 7 and at least one 3?

From 10 people, how many teams of 4 contain exactly one of Ana and Ben?

WEEK 29 WAYS OF COUNTING

Three Circles, In Symbols

Grade 5 filled the regions in. This week the picture becomes a formula.

IN CLASS THIS WEEK
Expressions and equivalent expressions (6.EE.2, 6.EE.3)

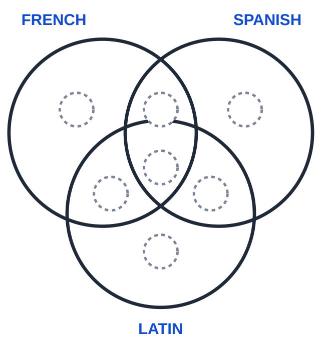
NUMBER WARM-UP · 2 MINUTES
22 + 18 + 15 = ? Is that more than the 40 students in the class?

WORDS TO KEEP

inclusion and exclusion add the parts, subtract the pairs, add back the triple

in symbols the same rule written once with letters, so it works for any numbers

Forty students. French 22, Spanish 18, Latin 15. Both French and Spanish 9, French and Latin 6, Spanish and Latin 5, and 3 take all three.



1 Fill the seven regions.

Start in the middle and work outward. Every number you were given already includes the regions inside it.

2 Then do it in symbols.

a. Add your seven regions. ________

b. Now the formula. With F, S and L for the three subjects, the number taking at least one is F + S + L − ______________________ + ______________________

c. Put the numbers in. ________ Same as your regions? ________ So ________ of the 40 take no language.

3 Why the middle comes back.

A student taking all three is counted ________ times by F + S + L, then removed ________ times by the three pairs. That leaves them counted ________ times, which is why the last term adds them back.

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 27: the formula written once with letters, and the middle-student argument for the last term. Grade 2 drew two circles. This is where it becomes algebra.

💬 **TALK ABOUT IT**

Write the two-circle version of the formula. Which terms disappear?

Guess the four-circle version. How many terms would it have, and what would the signs do?

STUCK? TRY THIS

Do the picture and the formula separately and only then compare. If they disagree, the picture is almost always right and the formula has a sign wrong.

★ **CHALLENGE ZONE**

Of 60 people, 30 like tea, 25 coffee, 20 juice, 12 tea and coffee, 9 tea and juice, 8 coffee and juice, 4 all three. How many like none?

Is it possible for the formula to give a number bigger than the group? What would that mean about the numbers you were given?

WEEK 30 RATIO & RATE

How Likely Is It?

Chance is a ratio: the outcomes you want against all the outcomes there are.

IN CLASS THIS WEEK
Ratio and rate reasoning (6.RP.1, 6.RP.3c)

NUMBER WARM-UP · 2 MINUTES
A bag has 3 red and 5 blue. What is the ratio of red to the whole bag?

WORDS TO KEEP

outcome one thing that could happen

chance as a ratio the outcomes you want, compared with all the outcomes, when every outcome is equally likely

A bag with **3 red**, **5 blue** and **2 green**. Ten marbles, and every one equally likely to be pulled out.

1 Write each as a ratio, then a percent.

Pulling out	wanted	out of	fraction	percent
a red	3	10		
a blue				
a green				
not a red				

2 Two checks.

a. Add the percents for red, blue and green. ________ Why must it come to that?

b. Red and not-red add to ________. That is Week 28 again, in percents.

3 Equally likely matters.

Somebody says a coin landing on its edge means heads, tails and edge are each one third. What is wrong with that?

✎ ADD TO YOUR RULE BOOK

Rule Book page 28: the bag, the four ratios, and the two checks. Chance done properly is a Grade 7 subject. This page only does the counting part, which is a ratio.

💬 TALK ABOUT IT

Why does the phrase **equally likely** have to be in the definition?

If the bag had 30 red, 50 blue and 20 green, would any of your answers change?

STUCK? TRY THIS

Count the outcomes you want and the outcomes there are, and write them as a ratio. Every question on this page is a counting question first.

★ CHALLENGE ZONE

A bag gives 40 percent chance of red and has 6 red marbles. How many marbles in total?

Two bags: 3 red of 10, and 5 red of 20. Which gives the better chance of red? Compare them as percents.

WEEK 31 DATA

Data Has a Shape

Nine numbers on a line. The shape says more than any single number can.

IN CLASS THIS WEEK
Statistical questions and distributions (6.SP.1, 6.SP.2)

NUMBER WARM-UP · 2 MINUTES
Is 'how tall am I' a statistical question? Is 'how tall are the students in my class'?

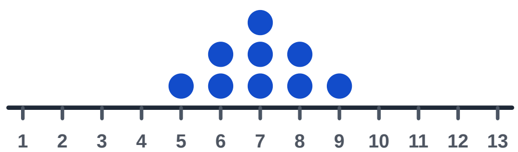
WORDS TO KEEP

statistical question one that expects the answers to vary

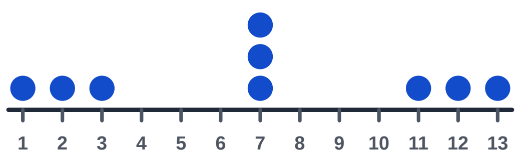
distribution the whole pattern of the values: where they cluster and how far they spread

Two classes, nine students each, books read last month.

Class A



Class B



1 Describe the two shapes.

a. Class A is bunched around ________

b. Class B is spread from ________ to ________, with a gap in the ________________________

2 Which questions are statistical?

Question	statistical?	why
How many books did Ana read?		
How many books did Class A read each?		
What is 7 × 8?		
How tall are sixth graders?		

3 A warning.

Somebody reports only that both classes averaged 7 books. Looking at the two pictures, what did that report throw away?

✎ ADD TO YOUR RULE BOOK

Rule Book page 29: both dot plots, one above the other on the same scale. The next two weeks are about what a single number does and does not tell you about these pictures.

TALK ABOUT IT

Why must a statistical question expect the answers to vary?

Which class would you rather teach, and what does the picture tell you that a single number cannot?

STUCK? TRY THIS

Draw both plots on the **same** scale or the comparison is meaningless. That is the commonest way a data picture misleads.

★ CHALLENGE ZONE

Collect nine numbers of your own and draw the dot plot. Describe its shape in one sentence.

Draw a plot of nine values that is bunched at both ends with nothing in the middle. What would that mean about the class?

WEEK 32 DATA

Center and Spread

One number for where the data sits, one for how far it wanders.

IN CLASS THIS WEEK
Measures of center and variability (6.SP.3, 6.SP.5)

NUMBER WARM-UP · 2 MINUTES
Put 5, 6, 6, 7, 7, 7, 8, 8, 9 in order. What is the middle one?

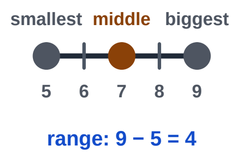
WORDS TO KEEP

mean add them all and divide by how many. The balance point

median the middle value once they are in order

range the biggest minus the smallest. One measure of spread

Class A again, on a line, with the smallest, middle and biggest marked.



1 Three numbers for Class A.

a. Mean: add them (________) and divide by 9. ________

b. Median: the 5th value in order. ________

c. Range: ________

2 Now Class B.

	mean	median	range
Class A			
Class B			

Which two columns match? ____________________ Which one does not? ____________________

3 What the mean does that the median does not.

Change the 9 in Class A to 90. Work out the new mean _______ and the new median _______. One moved a long way and one did not move at all. Say why.

____

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 30: the three numbers for both classes, and the one-value-changed experiment. A single outlier drags the mean and leaves the median where it was.

💬 **TALK ABOUT IT**

Which would you use to report a typical house price, and why?

Can the mean be a value that nobody in the data actually has?

STUCK? TRY THIS

Put the numbers in order before you do anything. The median needs it, and the range is then just the two ends.

★ **CHALLENGE ZONE**

Find a set of five numbers whose mean is 10 and whose median is 4.

Add one number to Class A so that the mean stays exactly 7. What must it be?

WEEK 33 DATA

Same Average, Different Story

Two classes, one average, and you would not want to swap them.

IN CLASS THIS WEEK
Summarizing data in context (6.SP.5, 6.SP.5c)

NUMBER WARM-UP · 2 MINUTES
Both classes average 7 books. Are they the same class?

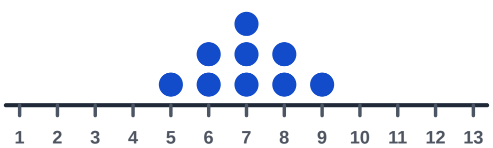
WORDS TO KEEP

summary one number standing in for many. Always a loss, sometimes a useful one

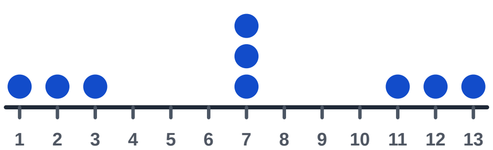
in context a number reported without saying what it measured and how it spread is not information

The two classes from Week 31, with their numbers from Week 32.

Class A



Class B



1 Same and different.

a. Both have a mean of ________ and a median of ________

b. Their ranges are ________ and ________

2 Write the report properly.

A summary should say what was measured, how many, the center, and the spread.

Class A:

Class B:

3 A headline.

A newspaper writes **Both classes read the same amount**. Is that true? Is it honest? They are not the same question.

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 31: both plots with all three numbers under each, and the headline with your verdict on it. Every summary throws something away, and this page is about saying what.

TALK ABOUT IT

Is the newspaper headline a lie? What word would you change?

Which single extra number would have made the headline honest?

STUCK? TRY THIS

A report needs four things: what was measured, how many there were, a center, and a spread. A report with only a center is not wrong, it is incomplete.

★ **CHALLENGE ZONE**

Find two sets of five numbers with the same mean, the same median and different ranges.

Find two sets with the same mean and the same range but obviously different shapes.

WEEK 34 DATA

Finding the Middle

The median needs the data sorted, and sorting has a price you already know.

IN CLASS THIS WEEK
Measures of center; problem solving (6.SP.3, 6.EE.2)

NUMBER WARM-UP · 2 MINUTES
To find the median of 99 numbers, which one do you need?

WORDS TO KEEP

cost the Grade 5 word. How many comparisons a method takes

do you need the whole order to find the middle you might not

A class of **99** numbers. You want the median, which is the 50th in order.

1 Read the ledger.

a. Sorting gets you the median, and much more besides. What else do you get?

____

b. Finding just the biggest costs 98. Does that suggest finding just the 50th should be cheaper than a full sort?________

2 Mean, median and mode, priced.

To find the	what you must do	roughly how much work
mean	add them all, divide	
median	sort, then look in the middle	
range	find the biggest and the smallest	

Which is cheapest? ______________________ Which is dearest? ______________________

3 Why the mean is cheap and fragile.

The mean costs one pass and moves a long way when one value changes. The median costs a sort and barely moves. Write one sentence about that trade.

____

✎ **ADD TO YOUR RULE BOOK**

Rule Book page 32: the three summary numbers with their costs beside them. Grade 5 priced methods and this page prices statistics, which is the same habit applied somewhere new.

💬 **TALK ABOUT IT**

Finding the biggest costs 98 and sorting costs thousands. Where does finding the 50th sit between them?

If you needed the median every day on new data, would you sort every time?

STUCK? TRY THIS

Ask what the summary actually requires. The mean needs every value once. The range needs two extremes. The median needs to know what is in the middle, which is more than two and less than everything.

★ **CHALLENGE ZONE**

Work out the cost of finding the mean, the median and the range of 1,000 numbers.

Describe a method for finding the biggest and the smallest together in fewer than $2n - 2$ comparisons. Pair the numbers up first.

Pigeonhole, Grown Up

Grade 3 put marbles in boxes. Now the boxes are a city.

IN CLASS THIS WEEK

Reasoning and expressions (6.EE.2, 6.EE.9)

NUMBER WARM-UP · 2 MINUTES

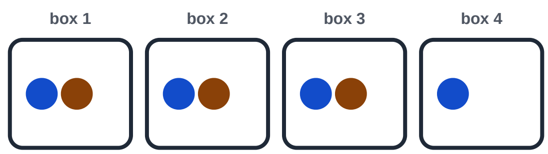
13 people, 12 birth months. Must two share a month?

WORDS TO KEEP

the pigeonhole principle more things than boxes means some box holds two or more

the hard part finding the boxes. The principle is easy and naming the boxes is not

The claim: somewhere in Los Angeles there are **two people with exactly the same number of hairs on their head**. Nobody has counted anybody.



7 things, 4 boxes

1 Name the things and the boxes.

a. The things being sorted are ______________________

b. The boxes are ______________________

c. A human head has fewer than 200,000 hairs, so there are at most ________ boxes.

d. Los Angeles has more than a million people. So ______________________

2 Push it further.

a. With a million people and 200,000 boxes, some box holds at least ________ people.

b. Write the general rule: with T things and B boxes, some box holds at least

____

3 Three more, and the boxes are the puzzle.

Claim	the things	the boxes
two people in a room of 400 share a birthday		
of any 5 numbers, two leave the same remainder after 4		
of any 6 people, three all know each other or three all do not		

The last one is much harder and is famous. You are not expected to finish it.

✎ ADD TO YOUR RULE BOOK

Rule Book page 33, the last new page: the hairs argument in four lines, and the general rule with T and B in it. Grade 3 met this with marbles. It has not changed at all, only the boxes got harder to find.

TALK ABOUT IT

The upper bound on hairs is doing all the work. What happens to the argument without it?

Which is harder: knowing the principle, or spotting what the boxes should be?

STUCK? TRY THIS

Say out loud what the things are and what the boxes are, in that order, before you count anything. If you cannot name the boxes, the principle cannot help you yet.

★ CHALLENGE ZONE

Twenty-three people. Show that two of them were born in the same month.

Pick any 10 numbers from 1 to 100. Show that two different groups of them have the same total. The boxes here are the possible totals.

WEEK 36 PATTERNS & RULES

The Rule Book, Finished

Thirty-three pages of rules. This week they become one book, and the line ends.

IN CLASS THIS WEEK
Review of the year

NUMBER WARM-UP · 2 MINUTES
No warm-up. Get every Rule Book page out and put them in order.

Every page should have a **rule** and a note saying whether you looked, checked or proved it. A rule with no such note is not finished.

1 The check list

Weeks	The page should say	✓
1 to 3	a letter for any number; shapes into rules; triangular numbers	
4 to 6	the handshake formula proved; pairing; the gnomon square	
7 to 9	looking back or jumping ahead; 1, 2, 4, 8, 16, 31; picture proofs	
10 to 12	proof by cases; double counting; six rules and how sure	
14 to 18	ratio as a rule; unit rate; percent; scaling; round robins	
19 to 22	the prime table; four quadrants; taxicab distance and its circle	
23 to 24	nets and counts; corners minus edges plus faces	
26 to 29	n choose k; the triangle explained; at least one; three circles	
30 to 34	chance as a ratio; shape, center and spread; what they cost	
35	pigeonhole, and finding the boxes	

2 Four last rules.

Write each one and mark it L, C or P.

a. The number of handshakes among n people. ____________________

b. The sum 1 + 2 + ... + n. ____________________

c. The sum of the first n odd numbers. ____________________

d. The number of regions in a circle with n points joined. ____________________

3 Teach one page.

Pick the proof you are proudest of and give it out loud to somebody who has not seen it. Write down the first thing they did not believe.

____

✎ ADD TO YOUR RULE BOOK

Write a first page for the book: your name, the year, and one sentence saying what it takes to turn a pattern into a rule. Then number every page and make a contents list.

TALK ABOUT IT

Five years of these books asked five questions: what is this, how many, can it be done, what does it cost, and what is the rule. Which one do you reach for first now?

Week 8 is the only page in the book about being wrong. Should there be more?

STUCK? TRY THIS

A page you cannot say out loud, with its reason, is not finished. That has been the test every year and it is the test this week.

★ CHALLENGE ZONE

Look back at the handshake problem: Grade 2 counted it, Grade 3 found the rule, Grade 4 used it, and Week 4 proved it. Write the four stages out. That is what this whole line of books has been doing.

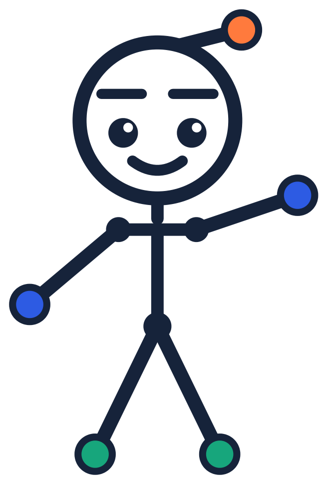
Find a pattern nobody has shown you, write the rule, and prove it. Put it on the last page. It is your book now.

CERTIFICATE OF COMPLETION

Rule Maker

awarded to

for a whole year of writing rules with letters in them, pairing the ends, building squares out of odd numbers, proving the handshakes at last, doubting 1, 2, 4, 8, 16, and, hardest of all, *saying how sure you are*.



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